Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE 2010

MARKING SCHEME

MATHEMATICS

HIGHER LEVEL

Contents

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GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2, …etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **(5, 5) marks** **Att (2, 2)**

1. **(a)** $x^2 - 6x + t = (x+k)^2$, where t and k are constants.
Find the value of k and the value of t.

(a) Equating coefficients **5 marks** **Att 2**
Values **5 marks** **Att 2**

1 (a)

$$x^2 - 6x + t = (x+k)^2 \;\Rightarrow\; x^2 - 6x + t = x^2 + 2kx + k^2.$$

$$\therefore\; 2k = -6 \text{ and } t = k^2 \;\Rightarrow\; k = -3 \text{ and } t = 9.$$

Or

(a) Perfect square **5 marks** **Att 2**
Values **5 marks** **Att 2**

1(a)

$x^2 - 6x + t = (x+k)^2$

$(x^2 - 6x + t)$ is a perfect square

$(x-3)^2 = x^2 - 6x + 9$

$\Rightarrow k = -3$ and $t = 9$

Blunders (-3)

B1 Expansion $(x+a)^2$ once only
B2 Not like-to-like in equating coefficients
B3 Indices

Part (b)	20 (5, 10, 5) marks	Att (2, 3, 2)

(b) Given that p is a real number, prove that the equation $x^2-4px-x+2p=0$ has real roots.

(b) Equation arranged	**5 marks**	**Att 2**
Correct substitution in b^2-4ac	**10 marks**	**Att 3**
Finish	**5 marks**	**Att 2**

1 (b) $x^2-4px-x+2p=0 \Rightarrow x^2+x(-4p-1)+2p=0$.

$b^2-4ac=(-4p-1)^2-4(2p)=16p^2+8p-8p+1=16p^2+1\geq 0$ for all p.

$\therefore$ Roots are real.

Blunders (-3)

- B1 Expansion of $(a+b)^2$ once only
- B2 Incorrect value a
- B3 Incorrect value b
- B4 Incorrect value c
- B5 Inequality sign
- B6 Indices
- B7 Incorrect deduction or no deduction

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

(c) $(x-2)$ and $(x+1)$ are factors of x^3+bx^2+cx+d.

(i) Express c in terms of b.

(ii) Express d in terms of b.

(iii) Given that b, c and d are three consecutive terms in an arithmetic sequence, find their values.

***f*(*2*) and *f*(-*1*)**	**5 marks**	**Att 2**
c* in terms of *b	**5 marks**	**Att 2**
d* in terms of *b	**5 marks**	**Att 2**
Values	**5 marks**	**Att 2**

1 (c) (i)

$(x-2)$ is a factor $\Rightarrow f(2)=0. \quad \therefore \quad 8+4b+2c+d=0 \quad \Rightarrow \quad 4b+2c+d=-8.$

$(x+1)$ is a factor $\Rightarrow f(-1)=0. \quad \therefore \quad -1+b-c+d=0 \quad \Rightarrow \quad b-c+d=1.$

$\therefore \quad 3b+3c=-9 \quad \Rightarrow \quad b+c=-3 \quad \Rightarrow \quad c=-b-3.$

1 (c) (ii) By part **(i)**

$$4b+2c+d=-8$$
$$\underline{2b-2c+2d=2}$$
$$6b+3d = -6 \quad \Rightarrow \quad 2b+d=-2 \quad \Rightarrow \quad d=-2b-2.$$

1 (c) (iii) An arithmetic sequence $b, c, d \quad \Rightarrow \quad c-b=d-c \Rightarrow 2c=b+d.$

$\therefore -2b-6=b-2b-2 \quad \Rightarrow \quad b=-4.$

$\therefore \quad c=1$ and $d=6$.

Blunders (-3)
B1 Indices
B2 Deduction root from factor
B3 Statement of AP

Slips (-1)
S1 Numerical

Worthless
W1 Geometric Sequence

Or

Division & remainder = 0	**5 marks**	**Att 2**
c* in terms of *b	**5 marks**	**Att 2**
d* in terms of *b	**5 marks**	**Att 2**
Values	**5 marks**	**Att 2**

1 (c) (i)

$$(x-2)(x+1)=(x^2-x-2) \quad \text{factor}$$

1 (c) (ii)

$$\begin{array}{r} x+(b+1) \\ x^2-x-2\overline{\big)\, x^3+bx^2+cx+d} \\ \underline{x^3-x^2-2x} \\ (b+1)x^2+(c+2)x+d \\ \underline{(b+1)x^2-(b+1)x-2(b+1)} \\ (c+2)x+(b+1)x+d+2(b+1)=0 \end{array}$$

since (x^2-x-2) is a factor

$$[(c+2)+(b+1)]x+[d+2(b+1)]=(0)x+(0)$$

Equating Coefficients

(i) $b+c+3=0 \Rightarrow c=-3-b$

(ii) $d+2b+2=0 \Rightarrow d=-2b-2$

1 (c) (iii) As in previous solution

Blunders (-3)

B1 $(x-2)(x+1)$ once only

B2 Indices

B3 Not like-to-like when equating coefficients

Slips (-1)

S1 Not changing sign when subtracting

Attempts

A1 Any effort at division

Worthless

W1 Geometric sequence

Other linear factor & multiplication	**5 marks**	**Att 2**
c* in terms of *b	**5 marks**	**Att 2**
d* in terms of *b	**5 marks**	**Att 2**
Values	**5 marks**	**Att 2**

1 (c) (i) (ii)

$$(x-2)(x+1)=(x^2-x-2) \text{ factor}$$

$$(x^2-x-2)\left(x-\frac{d}{2}\right)=x^3+bx^2+cx+d$$

$$x^3-x^2-2x-\frac{dx^2}{2}+\frac{dx}{2}+d=x^3+bx^2+cx+d$$

$$x^3+\left(-\frac{d}{2}-1\right)x^2+\left(-2+\frac{d}{2}\right)x+d=x^3+(b)x^2+(c)x+(d)$$

Equating Coefficients

(i) : $-2+\dfrac{d}{2}=c$

$-4+d=2c$

(ii) **:** $-\dfrac{d}{2}-1=b$

$-d-2=2b$

$-2b-2=d$

Put this value of *d* into (i)

(i) $-4+(-2b-2)=2c$

$-4-2b-2=2c$

$-6-2b=2c$

$c=-3-b$

1 (c) (iii) As in previous solution

Blunders (-3)

B1 Indices

B2 $(x-2)(x+1)$ once only

B3 Not like to like when equating coefficients

Attempts

A1 Other factors not linear in (1) only

Worthless

W1 Geometric sequence

QUESTION 2

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 (5, 5) marks** **Att(2, 2)**

(a) Solve the simultaneous equations

$$2x + 3y = 0$$
$$x + y + z = 0$$
$$3x + 2y - 4z = 9.$$

(a) One unknown	**5 marks**	**Att 2**
Other values	**5 marks**	**Att 2**

2 (a)

$$4x + 4y + 4z = 0$$
$$3x + 2y - 4z = 9$$
$$\overline{7x + 6y = 9}$$
$$4x + 6y = 0$$
$$\overline{3x = 9 \quad \Rightarrow \quad x = 3. \ \therefore \ y = -2 \text{ and } z = -1.}$$

Blunders (-3)
B1 Multiplying one side of equation only
B2 Not finding 2nd value, having found 1st value
B3 Not finding 3rd value, having found other two

Slips (-1)
S1 Numerical
S1 Not changing sign when subtracting

Worthless
W1 Trial and error only

Part (b)	20(10, 10) marks	Att (3, 3)

(b) The equation $x^2 - 12x + 16 = 0$ has roots α^2 and β^2, where $\alpha > 0$ and $\beta > 0$.

(i) Find the value of $\alpha\beta$.

(ii) Hence, find the value of $\alpha + \beta$.

(b) (i) Value of $\alpha\beta$	**10 marks**	**Att 3**
(b) (ii) Value of $(\alpha + \beta)$	**10 marks**	**Att 3**

2 (b) (i)

$$\alpha^2\beta^2 = 16 \quad \Rightarrow \quad \alpha\beta = 4.$$

2 (b) (ii)

$$\alpha^2 + \beta^2 = 12 \text{ and } \alpha\beta = 4.$$

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = 12 + 8 = 20.$$

$$\therefore \ \alpha + \beta = \sqrt{20} = 2\sqrt{5}.$$

Blunders (-3)

B1 Indices
B2 Incorrect sum
B3 Incorrect product
B4 Incorrect statements
B5 Excess value each time

Slips (-1)

S1 Numerical

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

(c) (i) Prove that for all real numbers a and b,

$$a^2 - ab + b^2 \geq ab.$$

(ii) Let a and b be non-zero real numbers such that $a + b \geq 0$.

Show that $\frac{a}{b^2} + \frac{b}{a^2} \geq \frac{1}{a} + \frac{1}{b}$.

(c) (i)	**5 marks**	**Att 2**
(ii) Factors	**5 marks**	**Att 2**
Use of part (i)	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

2 (c) (i)

$$(a-b)^2 \geq 0 \quad \Rightarrow \quad a^2 - 2ab + b^2 \geq 0.$$

$$\therefore \quad a^2 - ab + b^2 \geq ab.$$

2 (c) (ii) $$\frac{a}{b^2} + \frac{b}{a^2} = \frac{a^3 + b^3}{a^2b^2} = \frac{(a+b)(a^2 - ab + b^2)}{a^2b^2}.$$

But $$\frac{(a+b)(a^2 - ab + b^2)}{a^2b^2} \geq \frac{ab(a+b)}{a^2b^2}, \text{ by part (i)}$$

$$\frac{ab(a+b)}{a^2b^2} = \frac{a+b}{ab} = \frac{a}{ab} + \frac{b}{ab} = \frac{1}{b} + \frac{1}{a}.$$

$$\therefore \quad \frac{a}{b^2} + \frac{b}{a^2} \geq \frac{1}{a} + \frac{1}{b}.$$

OR

2 (c) (ii)

$$\frac{a}{b^2} + \frac{b}{a^2} \geq \frac{1}{a} + \frac{1}{b}$$

Multiply across by a^2b^2, which is positive:

$$\Leftrightarrow a^3 + b^3 \geq ab^2 + ba^2$$

$$\Leftrightarrow (a+b)(a^2 - ab + b^2) \geq ab(a+b)$$

$$\Leftrightarrow a^2 - ab + b^2 \geq ab, \qquad \text{since } a + b \geq 0$$

true, by part **(i)**.

Blunders (-3)

B1 Expansion $(a-b)^2$ once only

B2 Factors $a^3 + b^3$

B3 Indices

B4 Inequality sign

B5 Incorrect deduction or no deduction

Slips (-1)
S1 Numerical

Attepmts
A1 $a^3 + b^3 = (a+b)(a^2 + b^2)$

Worthless
W1 Particular values

(c) (i)		**5 marks**	**Att 2**
(ii)	**Common denominator**	**5 marks**	**Att 2**
	Factorised	**5 marks**	**Att 2**
	Finish	**5 marks**	**Att 2**

2 (c) (i)

$$\left(a^2 - ab + b^2\right) \geq ab, \Leftrightarrow \left(a^2 - ab + b^2\right) - ab \geq 0.$$

$$\begin{aligned}\left(a^2 - ab + b^2\right) - ab &= a^2 - 2ab + b^2 \\ &= (a-b)^2 \\ &\geq 0\end{aligned}$$

2 (c) (ii)

$$\frac{a}{b^2} + \frac{b}{a^2} \geq \frac{1}{a} + \frac{1}{b}, \Leftrightarrow \left(\frac{a}{b^2} + \frac{b}{a^2}\right) - \left(\frac{1}{a} + \frac{1}{b}\right) \geq 0.$$

$$\begin{aligned}\left(\frac{a}{b^2} + \frac{b}{a^2}\right) - \left(\frac{1}{a} + \frac{1}{b}\right) &= \frac{a^3 + b^3 - ab^2 - a^2b}{a^2b^2} \\ &= \frac{\left(a^3 - a^2b\right) - \left(ab^2 - b^3\right)}{a^2b^2} \\ &= \frac{a^2(a-b) - b^2(a-b)}{a^2b^2} \\ &= \frac{(a-b)\left[a^2 - b^2\right]}{a^2b^2} \\ &= \frac{(a-b)[(a-b)(a+b)]}{(ab)^2} \\ &= \frac{(a-b)^2(a+b)}{(ab)^2} \geq 0, \quad \text{since } a+b \geq 0\end{aligned}$$

Blunders (-3)
B1 Indices
B2 Inequality Sign
B3 Factors $\left(a^2 - b^2\right)$ once only
B4 Incorrect deduction or no deduction

Worthless
W1 Particular values

QUESTION 3

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att(2, 2)**

(a) Find x and y such that

$$\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 32 \end{pmatrix}.$$

Inverse of *A* evaluated	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

3 (a)

$$\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 32 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 20 \\ 32 \end{pmatrix}.$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{18-20}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 20 \\ 32 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -8 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Or

One unknown	**5 marks**	**Att 2**
Other unknown	**5 marks**	**Att 2**

3 (a)

(i) $3x + 4y = 20.6 \Rightarrow 18x + 24y = 120$

(ii) $5x + 6y = 32.4 \Rightarrow 20x + 24y = 128$

$$-2x = -8$$
$$x = 4$$

(i) $3x + 4y = 20$

$12 + 4y = 20$

$4y = 8 \Rightarrow y = 2$

Blunders (-3)

B1 Formula for inverse

B2 Matrix multiplication

Slips (-1)

S1 Each incorrect element in matrix multiplication

S2 Numerical

S3 Not changing sign when subtracting

Part (b) **20 (5, 5, 10) marks** **Att (2, 2, 3)**

(b)	Let $z_1 = s + 8i$ and $z_2 = t + 8i$, where $s \in \mathbb{R}, t \in \mathbb{R}$ and $i^2 = -1$.
(i)	Given that $\lvert z_1 \rvert = 10$, find the values of s.
(ii)	Given that $\arg(z_2) = \frac{3\pi}{4}$, find the value of t.

(b) (i) Values for modulus	**5 marks**	**Att 2**
Values of *s*	**5 marks**	**Att 2**
(ii) Value of *t*	**10 marks**	**Att 3**

3 (b) (i) $\lvert s + 8i \rvert = 10 \Rightarrow \sqrt{s^2 + 64} = 10 \Rightarrow s^2 = 36. \therefore s = \pm 6.$

3 (b) (ii) $\tan \frac{3\pi}{4} = \frac{8}{t} \Rightarrow -t = 8 \Rightarrow t = -8.$

Or

3 (b) (i) $z_1 = s + 8i \Rightarrow \lvert z_1 \rvert = 10$

$$\sqrt{s^2 + 64} = 10$$

$$s^2 + 64 = 100$$

$$s^2 = 36$$

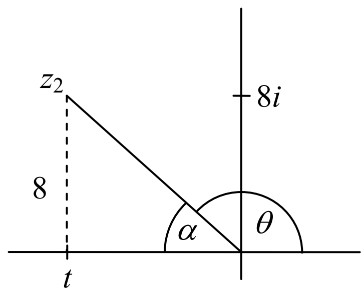
$$s = \pm 6$$

3 (b) (ii)

$$\tan \alpha = \tan \frac{\pi}{4} = 1$$

$$\Rightarrow \frac{8}{\lvert t \rvert} = 1$$

$$\lvert t \rvert = 8 \Rightarrow t = -8$$



$$\theta = \frac{3\pi}{4} \Rightarrow \alpha = \frac{\pi}{4}$$

Blunders (-3)

B1 Formula for modulus
B2 Indices
B3 Only one value for s
B4 Diagram for z_2 once only
B5 Incorrect argument
B6 Trig Definition
B7 Mod Values
B8 $\tan \frac{3\pi}{4} = 1$

Slips (-1)

S1 Trig value
S2 Numerical

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

(c) (i) Use De Moivre's theorem to find, in polar form, the five roots of the equation

$$z^5 = 1.$$

(ii) Choose one of the roots w, where $w \neq 1$. Prove that $w^2 + w^3$ is real.

(c) (i)	$z = cis\frac{2n\pi}{5}$	**5 marks**	**Att 2**
	Five roots	**5 marks**	**Att 2**
(c) (ii)	$w^2 + w^3$ **as sum of cos and sin**	**5 marks**	**Att 2**
	Show real	**5 marks**	**Att 2**

3 (c) (i)

$$z = (\cos 0 + i\sin 0)^{\frac{1}{5}} = \cos\left(\frac{0+2n\pi}{5}\right) + i\sin\left(\frac{0+2n\pi}{5}\right), \text{ for } n = 0, 1, 2, 3, 4.$$

$n = 0 \Rightarrow z_0 = 1.$

$n = 1 \Rightarrow z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}.$

$n = 2 \Rightarrow z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}.$

$n = 3 \Rightarrow z_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}.$

$n = 4 \Rightarrow z_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}.$

3 (c) (ii)

Let $w = z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

$$\therefore w^2 + w^3 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^2 + \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3$$

$$= \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$

$$= \left(\cos\frac{6\pi}{5} + \cos\frac{4\pi}{5}\right) + i\left(\sin\frac{6\pi}{5} + \sin\frac{4\pi}{5}\right)$$

$$= \left(2\cos\pi\cos\frac{\pi}{5}\right) + i\left(2\sin\pi\cos\frac{\pi}{5}\right)$$

$$= -2\cos\frac{\pi}{5} + i(0),$$

$$= -2\cos\frac{\pi}{5}, \quad \text{which is real}$$

Blunders (-3)

B1 Formula De Moivre once only
B2 Application De Moivre
B3 Indices

B4 Trig Formula
B5 Polar formula once only
B6 i

Slips (-1)
S1 Trig value
S2 Root omitted

Note: Must show (0)i

Attempt
A1 Use of decimals in c(ii)

Worthless
W1 $w=1$ used in c(ii)

QUESTION 4

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (c)	**25 (5, 5, 5, 5, 5) marks**	**Att (2, 2, 2, 2, 2)**

Part (a)	**10 (5, 5)marks**	**Att (2, 2)**

(a) Write the recurring decimal 0·474747….. as an infinite geometric series and hence as a fraction.

(a) Series	**5 marks**	**Att 2**
Fraction	**5 marks**	**Att 2**

4 (a)

$$0\cdot474747........ = \frac{47}{100} + \frac{47}{100^2} + \frac{47}{100^3} + ...$$

$$= \frac{a}{1-r} = \frac{\frac{47}{100}}{1-\frac{1}{100}} = \frac{47}{99}.$$

Blunders (-3)

B1 Infinity formula once only
B2 Incorrect *a*
B3 Incorrect *r*

Slips (-1)

S1 Numerical

Part (b)	15 (5, 5, 5) marks	Att (2, 2, 2)

(b) In an arithmetic sequence, the fifth term is –18 and the tenth term is 12.
(i) Find the first term and the common difference.
(ii) Find the sum of the first fifteen terms of the sequence.

(b) (i)	**Terms in *a* and *d***	**5 marks**	**Att 2**
	Values of *a* and *d*	**5 marks**	**Att 2**
(b) (ii)	**Sum**	**5 marks**	**Att 2**

4 (b) (i)

$$T_5 = -18 \quad \Rightarrow \quad a + 4d = -18$$
$$T_{10} = 12 \quad \Rightarrow \quad \underline{a + 9d = 12}$$
$$-5d = -30 \quad \Rightarrow \quad d = 6 \text{ and } a = -42$$

4 (b) (ii)

$$S_n = \frac{n}{2}\{2a + (n-1)d\}. \quad \therefore \; S_{15} = \frac{15}{2}\{-84 + 14(6)\} = \frac{15}{2}(0) = 0.$$

Blunders (-3)
B1 Term of A.P.
B2 Formula A.P. once only (term)
B3 Incorrect *a*
B4 Incorrect *d*
B5 Formula for sum arithmetic series once only

Slips (-1)
S1 Numerical

Worthless
W1 Treats as G.P.

Part (c) **25 (5, 5, 5, 5, 5) marks** **Att (2, 2, 2, 2, 2)**

(c)	**(i)**	Show that $(r+1)^3-(r-1)^3=6r^2+2$.
	(ii)	Hence, or otherwise, prove that $\sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$.
	(iii)	Find $\sum_{r=11}^{30}(3r^2+1)$.

(c) (i) **5 marks** **Att 2**

4 (c) (i) $(r+1)^3-(r-1)^3=r^3+3r^2+3r+1-(r^3-3r^2+3r-1)=6r^2+2.$

OR

4 (c) (i)

$$\begin{aligned}(r+1)^3-(r-1)^3 &= [(r+1)-(r-1)][(r+1)^2+(r+1)(r-1)+(r-1)^2]\\ &= [r+1-r+1][r^2+2r+1+r^2-1+r^2-2r+1]\\ &= (2)(3r^2+1)\\ &= 6r^2+2\end{aligned}$$

Blunders (-3)

B1 Expansion of $(r+1)^3$ once only

B2 Expansion of $(r-1)^3$ once only

B3 Formula a^3-b^3

B4 Indices

B5 Expansion of $(r+1)^2$ once only

B6 Expansion of $(r-1)^2$ once only

B7 Binomial expansion once only

(c) (ii) Cancellation & addition shown **5 marks** **Att 2**
Prove **5 marks** **Att 2**

4 (c) (ii)

$$\cancel{2^3} - 0^3 = 6(1^2) + 2$$
$$\cancel{3^3} - 1^3 = 6(2^2) + 2$$
$$\cancel{4^3} - \cancel{2^3} = 6(3^2) + 2$$
$$\vdots$$
$$\vdots$$
$$\cancel{(n-1)^3} - \cancel{(n-3)^3} = 6(n-2)^2 + 2$$
$$n^3 - \cancel{(n-2)^3} = 6(n-1)^2 + 2$$
$$(n+1)^3 - \cancel{(n-1)^3} = 6n^2 + 2$$

$$(n+1)^3 + n^3 - 1 = 6\sum_{r=1}^{n} r^2 + 2n$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}\left(n^3 + 3n^2 + 3n + 1 + n^3 - 1 - 2n\right) = \frac{1}{6}\left(2n^3 + 3n^2 + n\right)$$

$$= \frac{n\left(2n^2 + 3n + 1\right)}{6} = \frac{n(n+1)(2n+1)}{6}.$$

OR

4 (c) (ii) Prove by induction that $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2 = \frac{n}{6}(n+1)(2n+1)$

P(1): Test $n = 1$: $\frac{1}{6}(2)(3) = 1 \Rightarrow$ True for $n = 1$.

P(k): Assume true for $n = k$: $\Rightarrow S_k = \frac{k}{6}(k+1)(2k+1)$

To prove : $S_{k+1} = \frac{k+1}{6}(k+2)(2k+3)$

Proof: $S_{k+1} = 1^2 + 2^2 + \ldots\ldots\ldots k^2 + (k+1)^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$, using P(k)

$$= \frac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right]$$
$$= \frac{(k+1)}{6}\left[2k^2 + k + 6k + 6\right]$$
$$= \frac{k+1}{6}\left[2k^2 + 7k + 6\right]$$
$$= \frac{k+1}{6}\left[(k+2)(k+3)\right]$$

$\Rightarrow$ Formula true for $n = (k+1)$ if true for $n = k$
It is true for $n = 1 \Rightarrow$ true for all n

* Must show three terms at start and two at finish or vice versa in first method.

Blunders (-3)
B1 Indices
B2 Cancellation must be shown or implied
B3 Term omitted
B4 Expansion $(n+1)^3$ once only

(c) (iii) Substitution of *r* = 30 and *r* = 10 — 5 marks — Att 2
Sum — 5 marks — Att 2

4 (c) (iii)

$$\sum_{r=11}^{30}\left(3r^2+1\right)=3\sum_{1}^{30}r^2-3\sum_{1}^{10}r^2+30-10$$

$$=\frac{3(30)(31)(61)}{6}-\frac{3(10)(11)(21)}{6}+20=28365-1155+20=27230.$$

Blunders (-3)
B1 Formula
B2 Not (**Σ**30 – **Σ**10)
B3 Value *n*

Slips (-1)
S1 Numerical

QUESTION 5

Part (a)	**10 (5, 5)marks**	**Att(2, 2)**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5)marks**	**Att (2, 2)**

(a) Solve $\log_2(x+6) - \log_2(x+2) = 1$.

(a) Log law applied	**5 marks**	**Att 2**
Value	**5 marks**	**Att 2**

$$\log_2(x+6) - \log_2(x+2) = 1.$$

$$\therefore \log_2\left(\frac{x+6}{x+2}\right) = 1 \Rightarrow \frac{x+6}{x+2} = 2$$

$$\therefore 2x + 4 = x + 6 \Rightarrow x = 2.$$

Blunders (-3)

B1 Log laws

B2 Indices

Part (b)	20 (5, 5, 10) marks	Att (2, 2, 3)

(b) Use induction to prove that

$$2+(2\times 3)+(2\times 3^2)+(2\times 3^3)+.........+(2\times 3^{n-1}) = 3^n-1,$$

where n is a positive integer.

Part (b)			
Part (b)	***P*(1)**	**5 marks**	**Att 2**
	***P*(*k*)**	**5 marks**	**Att 2**
	***P*(*k* + 1)**	**10 marks**	**Att 3**

5 (b)

Test for $n = 1$, $P(1)=3^1-1=2$.

$\therefore$ True for $n = 1$.

Assume $P(k)$. (That is, assume true for $n = k$.).

i.e., assume $S_k=3^k-1$, where S_k is the sum of the first k terms.

Deduce $P(k+1)$. (That is, deduce truth for $n = k+1$.)

i.e. deduce that $S_{k+1}=3^{k+1}-1$.

Proof: $S_{k+1}=S_k+T_{k+1}=3^k-1+2\times 3^k=3(3^k)-1=3^{k+1}-1$.

$\therefore$ True for $n = k+1$.

So, $P(k+1)$ is true whenever $P(k)$ is true. Since $P(1)$ is true, then, by induction, $P(n)$ is true, for all positive integers n.

Blunders (-3)

B1 Indices

B2 Not T_{k+1} added to each side

B3 Not $n=1$

Worthless

W1 $P(0)$

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

(c) (i) Expand $\left(x+\frac{1}{x}\right)^2$ and $\left(x+\frac{1}{x}\right)^4$.

(ii) Hence, or otherwise, find the value of $x^4+\frac{1}{x^4}$, given that $x+\frac{1}{x}=3$.

(c) (i) $\left(x+\frac{1}{x}\right)^2$	**5 marks**	**Att 2**
$\left(x+\frac{1}{x}\right)^4$	**5 marks**	**Att 2**
(c) (ii) Terms collected	**5 marks**	**Att 2**
Value	**5 marks**	**Att 2**

5 (c) (i)

$$\left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2}.$$

$$\left(x+\frac{1}{x}\right)^4 = x^4+{}^4C_1x^3\left(\frac{1}{x}\right)+{}^4C_2x^2\left(\frac{1}{x}\right)^2+{}^4C_3x\left(\frac{1}{x}\right)^3+\left(\frac{1}{x}\right)^4$$

$$= x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}.$$

OR

$$\left(x+\frac{1}{x}\right)^4 = \left[\left(x+\frac{1}{x}\right)^2\right]^2$$

$$= \left[\left(x^2+\frac{1}{x^2}\right)+2\right]^2$$

$$= \left(x^2+\frac{1}{x^2}\right)^2+2(2)\left(x^2+\frac{1}{x^2}\right)+4$$

$$= x^4+2+\frac{1}{x^4}+4x^2+\frac{4}{x^2}+4$$

$$= x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$$

5 (c) (ii)

$$\left(x+\frac{1}{x}\right)^4 = 81 = x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4} = \left(x^4+\frac{1}{x^4}\right)+4\left(x^2+\frac{1}{x^2}\right)+6$$

$$\therefore\ x^4+\frac{1}{x^4} = 75-4\left(x^2+\frac{1}{x^2}\right).$$

But $x^2+2+\frac{1}{x^2}=9 \ \Rightarrow\ x^2+\frac{1}{x^2}=7.$

$$\therefore\ x^4+\frac{1}{x^4} = 75-28 = 47.$$

Blunders (-3)

B1 Binomial Expansion once only

B2 Indices

B3 Value $\binom{n}{r}$ or no $\binom{n}{r}$

B4 $x^0 \neq 1$

B5 Expansion $\left(x+\frac{1}{x}\right)^2$ once only

B6 Expansion $\left(x+\frac{1}{x}\right)^4$ once only

B7 Value $\left(x^2+\frac{1}{x^2}\right)$ or no value $\left(x^2+\frac{1}{x^2}\right)$

OR

(c) (ii) Roots	**5 marks**	**Att 2**
Value	**5 marks**	**Att 2**

5 (c) (ii)

$$\left(x+\frac{1}{x}\right)^2=(3)^2$$

$$x^4-7x^2+1=0$$

$$x^2=\frac{7\pm 3\sqrt{5}}{2}$$

$$x^4+\frac{1}{x^4}=\left(\frac{7+3\sqrt{5}}{2}\right)^2+\left(\frac{2}{7+3\sqrt{5}}\right)^2$$

$$=\frac{94+42\sqrt{5}}{4}+\frac{4}{94+42\sqrt{5}}$$

$$=\frac{2209+987\sqrt{5}}{47+21\sqrt{5}}\cdot\frac{47-21\sqrt{5}}{47-21\sqrt{5}}$$

$$=\frac{103823+46389\sqrt{5}-46389\sqrt{5}-103635}{2209-2205}$$

$$=47$$

Similarly, when $x^2=\frac{7-3\sqrt{5}}{2}$, $x^4+\frac{1}{x^4}=47$.

Note: must test two roots.

Blunders (-3)

B1 Roots formula once only

B2 Indices

B3 Expansion $\left(x+\frac{1}{x}\right)^2$ once only

Attempts

A1 Decimals used

QUESTION 6

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

(a) The equation $x^3 + x^2 - 4 = 0$ has only one real root.
Taking $x_1 = \frac{3}{2}$ as the first approximation to the root, use the Newton-Raphson method to find x_2, the second approximation.

(a) Differentiation	**5 marks**	**Att 2**
Value	**5 marks**	**Att 2**

6 (a)

$$x_2 = f\left(\frac{3}{2}\right) - \frac{f\left(\frac{3}{2}\right)}{f'\left(\frac{3}{2}\right)}.$$

$$f(x) = x^3 + x^2 - 4 \quad \Rightarrow \quad f\left(\frac{3}{2}\right) = \frac{27}{8} + \frac{9}{4} - 4 = \frac{13}{8}.$$

$$f'(x) = 3x^2 + 2x \quad \Rightarrow \quad f'\left(\frac{3}{2}\right) = \frac{27}{4} + 3 = \frac{39}{4}.$$

$$\therefore \; x_2 = \frac{3}{2} - \frac{\frac{13}{8}}{\frac{39}{4}} = \frac{3}{2} - \frac{1}{6} = \frac{8}{6} = \frac{4}{3}.$$

Blunders (-3)
B1 Newton-Raphson formula once only
B2 Differentiation
B3 Indices
B4 $x_1 \neq \frac{3}{2}$

Part (b) **20 (5, 5, 10) marks** **Att (2, 2, 3)**

(b) Parametric equations of a curve are:

$$x = \frac{2t-1}{t+2}$$

$$y = \frac{t}{t+2}, \text{ where } t \in \mathbb{R} \setminus \{-2\}.$$

(i) Find $\frac{dy}{dx}$.

(ii) What does your answer to part **(i)** tell you about the shape of the graph?

(b)(i) $\frac{dx}{dt}$ **or** $\frac{dy}{dt}$ **5 marks** **Att 2**

$\frac{dy}{dx}$ **5 marks** **Att 2**

6 (b) (i)

$$x = \frac{2t-1}{t+2} \Rightarrow \frac{dx}{dt} = \frac{(t+2)2-(2t-1)1}{(t+2)^2} = \frac{5}{(t+2)^2}.$$

$$y = \frac{t}{t+2} \Rightarrow \frac{dy}{dt} = \frac{1(t+2)-t(1)}{(t+2)^2} = \frac{2}{(t+2)^2}.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2}{(t+2)^2}.\frac{(t+2)^2}{5} = \frac{2}{5}.$$

OR

(b) (i) Elimination of *t* **5 marks** **Att 2**

$\frac{dy}{dx}$ **5 marks** **Att 2**

6 (b) (i)

$$x = \frac{2t-1}{t+2} \qquad y = \frac{t}{t+2}$$

$$\Rightarrow t = \frac{(-2x-1)}{(x-2)} \qquad t = \frac{-2y}{y-1}$$

$$t = \frac{(-2x-1)}{(x-2)} = \frac{(-2y)}{(y-1)}$$

$$\Rightarrow 2x+1 = 5y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{5}$$

Blunders (-3)

B1 Indices

B2 Differentiation

B3 Incorrect $\frac{dy}{dx}$

Attempts

A1 Error in differentiation formula

(b) (ii)	**10 marks**	**Att 3**

6 (b) (ii)
Since the slope is constant, it is a (subset of a) straight line.

If "line" is not mentioned in the answer, can only get Att 3 at most.

Part (c)	**20(5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

(c) A curve is defined by the equation $x^2y^3 + 4x + 2y = 12.$

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) Show that the tangent to the curve at the point $(0, 6)$ is also the tangent to it at the point $(3, 0)$.

(c) (i)	**Differentiation**	**5 marks**	**Att 2**
	Isolate $\frac{dy}{dx}$	**5 marks**	**Att 2**
(c) (ii)	**Equation 1st Tangent**	**5 marks**	**Att 2**
	Equation 2nd Tangent	**5 marks**	**Att 2**

6 (c) (i)

$$x^2y^3 + 4x + 2y = 12 \quad \Rightarrow \quad x^2.3y^2\frac{dy}{dx} + y^3.2x + 4 + 2\frac{dy}{dx} = 0.$$

$$\therefore \quad \frac{dy}{dx}\left(3x^2y^2 + 2\right) = -2xy^3 - 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2xy^3 - 4}{3x^2y^2 + 2}.$$

6 (c) (ii)

$$\frac{dy}{dx} = \frac{-2xy^3 - 4}{3x^2y^2 + 2}$$

Slope of tangent at (0, 6) is $\frac{-4}{2} = -2$.

Equation of tangent at (0, 6) is $y - 6 = -2x \quad \Rightarrow \quad 2x + y = 6$.

Slope of tangent at (3, 0) is $\frac{-4}{2} = -2$.

Equation of tangent at (3, 0) is $y = -2(x - 3) \quad \Rightarrow \quad 2x + y = 6$.

$\therefore$ same tangent.

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Incorrect value of x or no value of x in slope
B4 Incorrect value of y or no value of y in slope
B5 Equation of tangent
B6 Incorrect conclusion or no conclusion

Slips (-1)
S1 Numerical

Attempts
A1 Error in differentiation formula
A2 $\frac{dy}{dx} = 3x^2y^2\frac{dy}{dx} + 4 + 2\frac{dy}{dx} \rightarrow$ and uses the three $\left(\frac{dy}{dx}\right)$ term

OR

(c) (ii) **10 marks** **Att 3**

6 (c) (ii)

$$\frac{dy}{dx} = \frac{-2xy^3-4}{3x^2y^2+2}$$

Slope of tangent at $A(0, 6)$ is $\frac{-4}{2} = -2 = m_1$

Slope of tangent at $B(3, 0)$ is $\frac{-4}{2} = -2 = m_2$

Slope of the line $[AB]$ is $m_3 = \frac{-6}{3} = -2$

So, $m_1 = m_2 = m_3 = -2$

$\Rightarrow$ the line through A and B is the tangent at both points.

Blunders (-3)

B1 Slope omitted

B2 Incorrect deduction or no deduction

QUESTION 7

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

(a) Differentiate x^2 with respect to x from first principles.

$f(x+h)-f(x)$ **simplified**	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

7 (a)

$$f(x)=x^2 \Rightarrow f(x+h)=(x+h)^2.$$

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Limit}}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$=\lim_{h\to 0}(2x+h)=2x.$$

Blunders (-3)

B1 $f(x+h)$

B2 Indices

B3 Expansion of $(x+h)^2$ once only

B4 $h\to\infty$

B5 No limits shown or implied or no indication of $h\to 0$

Part (b) **20(10, 10) marks** **Att (3, 3)**

(b) Let $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

(i) Find $\dfrac{dy}{dx}$.

(ii) Show that $\dfrac{dy}{dx} = 1 + y^2$.

(b) (i) Differentiation **10 marks** **Att 3**
(ii) Show **10 marks** **Att 3**

7 (b) (i)

$$y = \frac{\cos x + \sin x}{\cos x - \sin x} \Rightarrow \frac{dy}{dx} = \frac{(\cos x - \sin x)(-\sin x + \cos x) - (\cos x + \sin x)(-\sin x - \cos x)}{(\cos x - \sin x)^2}.$$

$$\frac{dy}{dx} = \frac{(\cos x - \sin x)^2 + (\cos x + \sin x)^2}{(\cos x - \sin x)^2} = \frac{2}{(\cos x - \sin x)^2}.$$

7 (b) (ii)

$$\frac{dy}{dx} = \frac{(\cos x - \sin x)^2 + (\cos x + \sin x)^2}{(\cos x - \sin x)^2} = 1 + \frac{(\cos x + \sin x)^2}{(\cos x - \sin x)^2} = 1 + y^2.$$

OR

7 (b) (i) & 7 (b) (ii)

$$y = \frac{\cos x + \sin x}{\cos x - \sin x} = (\cos x + \sin x).(\cos x - \sin x)^{-1}$$

$$\frac{dy}{dx} = (\cos x + \sin x)\left[-1.(\cos x - \sin x)^{-2}(-\sin x - cox)\right] + (\cos x - \sin x)^{-1}(-\sin x + \cos x)$$

$$= \frac{(\cos x + \sin x)^2}{(\cos x - \sin)^2} + \frac{\cos x - \sin x}{\cos x - \sin x}$$

$$= \left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^2 + 1$$

$$= y^2 + 1$$

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Trig formula

Attempts
A1 Error in differentiation Formula

Worthless
W1 Integration

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

(c) The function $f(x)=(1+x)\log_e(1+x)$ is defined for $x>-1$.

(i) Show that the curve $y=f(x)$ has a turning point at $\left(\frac{1-e}{e},-\frac{1}{e}\right)$.

(ii) Determine whether the turning point is a local maximum or a local minimum.

(c) (i) $f'(x)$	**5 marks**	**Att 2**
Value of x	**5 marks**	**Att 2**
Value of y	**5 marks**	**Att 2**
(c) (ii) Turning points	**5 marks**	**Att 2**

7 (c) (i)

$$f(x)=(1+x)\log_e(1+x) \quad \Rightarrow \quad f'(x)=(1+x).\left(\frac{1}{1+x}\right)+\log_e(1+x)=1+\log_e(1+x).$$

$$f'(x)=0 \quad \Rightarrow \quad \log_e(1+x)=-1 \quad \Rightarrow \quad 1+x=e^{-1}. \quad \therefore \quad x=\frac{1}{e}-1=\frac{1-e}{e}.$$

$$y=\left(\frac{1}{e}\right)\log_e\left(\frac{1}{e}\right)\Rightarrow y=\frac{1}{e}(-\log_e e)=-\frac{1}{e}.$$ So turning point is $\left(\frac{1-e}{e},\ -\frac{1}{e}\right)$.

OR

7 (c) (i) $f'(x)=[\log_e(1+x)]+1$

At $x=\frac{1-e}{e}$, $f'(x)=\log_e\left(1+\frac{1-e}{e}\right)+1=\log_e\left(\frac{e+1-e}{e}\right)+1=\log_e\left(\frac{1}{e}\right)+1$

$=\left[\log_e(1)-\log_e(e)\right]+1$

$=0-1+1 \quad =0.$

So $f'(x)=0$ at $x=\frac{1-e}{e}$.

Also, at $x=\frac{1-e}{e}$, $y=\left(\frac{1}{e}\right)\log_e\left(\frac{1}{e}\right)\Rightarrow y=\frac{1}{e}(-\log_e e)=-\frac{1}{e}$.

So turning point is $\left(\frac{1-e}{e},\ -\frac{1}{e}\right)$.

7 (c) (ii)

$$f''(x)=\frac{1}{1+x}\Rightarrow f''\left(\frac{1-e}{e}\right)=\frac{1}{1+\frac{1-e}{e}}=\frac{e}{1}=e>0. \quad \therefore \quad \left(\frac{1-e}{e},-\frac{1}{e}\right)$$ is a local minimum.

Blunders (-3)
B1 Differentiation
B2 $f'(x)\neq 0$
B3 Indices
B4 Incorrect deduction or no deduction

Slips (-1)
S1 $\log_e e\neq 1$

Attempts
A1 Error in differentiation formula

Worthless
W1 Integration

QUESTION 8

Part (a)	**10 marks**	**Att3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

Part (a) **10 marks** **Att 3**

(a) Find $\int \left(\sin 2x + e^{4x}\right) dx.$

(a) **10 marks** **Att 3**

8 (a)

$$\int \left(\sin 2x + e^{4x}\right) dx = -\frac{1}{2}\cos 2x + \frac{1}{4}e^{4x} + c$$

Blunders (-3)

B1 Integration

B2 No *'c'*

Attempts

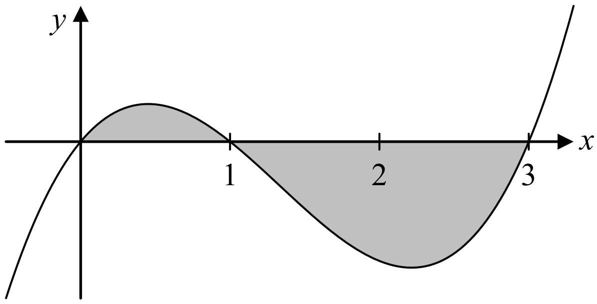
A1 Only *'c'* correct $\Rightarrow$ Att 3

Worthless

W1 Differentiation instead of integration

Part (b) **20 (10, 5, 5) marks** **Att (3,2,2)**

(b) The curve $y = 12x^3 - 48x^2 + 36x$ crosses the x-axis at $x = 0, x = 1$ and $x = 3$, as shown.



Calculate the total area of the shaded regions enclosed by the curve and the x-axis.

(b) First area	**5 marks**	**Att 2**
Second area	**5 marks**	**Att 2**
Total Area	**5 marks**	**Att 2**

8 (b)

Required area = $\left|\int_0^1 \left(12x^3 - 48x^2 + 36x\right) dx\right| + \left|\int_1^3 \left(12x^3 - 48x^2 + 36x\right) dx\right|$

$\left|\int_0^1 \left(12x^3 - 48x^2 + 36x\right) dx\right| = \left|3x^4 - 16x^3 + 18x^2\right|_0^1 = \left|3 - 16 + 18\right| = 5$.

$\left|\int_1^3 \left(12x^3 - 48x^2 + 36x\right) dx\right| = \left|3x^4 - 16x^3 + 18x^2\right|_1^3$

$= \left|(243 - 432 + 162) - (3 - 16 + 18)\right| = \left|-27 - 5\right| = 32$

$\therefore$ the required area is 5 + 32 = 37.

Blunders (-3)

B1 Integration
B2 Indices
B3 Error in area formula
B4 Incorrect order in applying limits
B5 Not calculating substituted limits
B6 Uses $\pi\int ydx$ for area formula

Attempts

A1 Uses volume formula
A2 Uses y^2 in formula

Worthless

W1 Wrong area formula and no work

Part (c) **20 (5, 5 10) marks** **Att (2, 2, 3)**

(c) **(i)** Find, in terms of a and b

$$I = \int_a^b \frac{\cos x}{1+\sin x}dx.$$

(ii) Find in terms of a and b

$$J = \int_a^b \frac{\sin x}{1+\cos x}dx.$$

(iii) Show that if $a + b = \frac{\pi}{2}$, then $I = J$.

(c) (i)	**5 marks**	**Att 2**
(ii)	**5 marks**	**Att 2**
(iii)	**10 marks**	**Att 2**

8 (c) (i)

$$I = \int_a^b \frac{\cos x}{1+\sin x}dx. \qquad \text{Let } u = 1+\sin x \quad \therefore\ du = \cos x dx.$$

$$I = \int_{1+\sin a}^{1+\sin b} \frac{du}{u} = \left[\log_e u\right]_{1+\sin a}^{1+\sin b} = \log_e(1+\sin b) - \log_e(1+\sin a).$$

$$I = \log_e\left(\frac{1+\sin b}{1+\sin a}\right).$$

8 (c) (ii)

$$J = \int_a^b \frac{\sin x}{1+\cos x}dx. \quad \text{Let } u = 1+\cos x \quad \therefore\ du = -\sin x dx.$$

$$J = \int_{1+\cos a}^{1+\cos b} \frac{-du}{u} = -\left[\log_e u\right]_{1+\cos a}^{1+\cos b} = -\log_e(1+\cos b) + \log_e(1+\cos a).$$

$$J = \log_e\left(\frac{1+\cos a}{1+\cos b}\right).$$

8 (c) (iii)

When $a + b = \frac{\pi}{2}$, then

$$I = \log_e\left(\frac{1+\sin b}{1+\sin a}\right) = \log_e\left(\frac{1+\sin\left(\frac{\pi}{2}-a\right)}{1+\sin\left(\frac{\pi}{2}-b\right)}\right) = \log_e\left(\frac{1+\cos a}{1+\cos b}\right) = J.$$

Blunders (-3)

B1 Integration
B2 Differentiation
B3 Trig Formula
B4 Logs
B5 Limits
B6 Incorrect order in applying limits
B7 Not calculating substituted limits
B8 Not changing limits
B9 Incorrect deduction or no deduction

Slips (-1)

S1 Numerical
S2 Trig value

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State Examinations Commission

LEAVING CERTIFICATE 2010

MARKING SCHEME

MATHEMATICS – PAPER 2

HIGHER LEVEL

GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2, …etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.

QUESTION 1

Part (a)	**10 marks**	**Att 3**
Part (b)	**15 (5, 10) marks**	**Att (2, 3)**
Part (c)	**25 (10, 15) marks**	**Att (3, 5)**

Part (a)	**10 marks**	**Att 3**

1 (a) A circle with centre $(3,-4)$ passes through the point $(7,-3)$.
Find the equation of the circle.

(a)	**10 marks**	**Att 3**

1 (a)

Centre is $(3,-4)$ and $r = \sqrt{(3-7)^2 + (-4+3)^2} = \sqrt{16+1} = \sqrt{17}$.

Circle : $(x-3)^2 + (y+4)^2 = 17$.

or

1 (a)

$x^2 + y^2 - 6x + 8y + c = 0$ But $(7,-3) \in$ Circle

$\Rightarrow 49 + 9 - 42 - 24 + c = 0 \Rightarrow c = 8$

Equation of circle $x^2 + y^2 - 6x + 8y + 8 = 0$

Blunders (-3)
B1 Error in substituting into distance formula
B2 Incorrect sign assigned to centre in equation of circle

Slips (-1)
S1 Arithmetic error

Attempts (3marks)
A1 Radius length
A2 Equation of circle without radius evaluated
A3 Equation of circle without substitution for c
A4 Substitution of (7,-3) and stops
A5 $x^2 + y^2 = 17$

Misreading(-1)
M1 (7,-3) as centre of circle

Part (b)	**15 (5, 10) marks**	**Att (2 , 3)**

1 (b) (i) Find the centre and radius of the circle

$$x^2 + y^2 - 8x - 10y + 32 = 0.$$

(ii) The line $3x + 4y + k = 0$ is a tangent to the circle $x^2 + y^2 - 8x - 10y + 32 = 0$. Find the two possible values of k.

(b)(i)	**5 marks**	**Att 2**

1 (b) (i) Centre is $(4, 5)$. $r = \sqrt{16 + 25 - 32} = \sqrt{9} = 3$.

or

1 (b) (i) $(x^2 - 8x + 16) + (y^2 - 10y + 25) = -32 + 16 + 25$

$(x-4)^2 + (y-5)^2 = 9$

Centre (4, 5) Radius = $\sqrt{9}$ or 3

Both correct	5 marks
One correct	2 marks
None correct	0 marks

(b) (ii)	**10 marks**	**Att 3**

1 (b) (ii)

$$\left|\frac{3(4) + 4(5) + k}{\sqrt{9+16}}\right| = 3 \Rightarrow |32 + k| = 15 \Rightarrow 32 + k = \pm 15.$$

$$32 + k = 15 \text{ or } 32 + k = -15 \Rightarrow k = -17 \text{ or } k = -47.$$

* Accept candidates centre and radius from (b)(i)

or

1 (b) (ii) $y = \dfrac{-3x - k}{4}$

$x^2 + \left(\dfrac{-3x-k}{4}\right)^2 - 8x - 10\left(\dfrac{-3x-k}{4}\right) + 32 = 0$

$25x^2 + (6k-8)x + k^2 + 40k + 512 = 0$

Equal roots $\Rightarrow$ $(6k-8)^2 = 100(k^2 + 40k + 512)$

$64k^2 + 4096k + 51136 = 0$

$k^2 + 64k + 799 = 0$

$(k + 17)(k + 47) = 0$ $k = -17$ and $k = -47$

Blunders (-3)

B1 Error in substitution into perpendicular distance formula
B2 One value of k only
B3 Incorrect squaring
B4 Error in factors

Slips (-1)

S1 Arithmetic error

Attempts (3marks)

A1 Some correct substitution into perpendicular formula

A2 Some correct substitution of either x or y from linear equation into circle

Part (c)	**25 (10, 15) marks**	**Att (3, 5)**

1 (c) A circle has the line $y = 2x$ as a tangent at the point $(2, 4)$. The circle also contains the point $(4, -2)$. Find the equation of the circle.

(c) First equation in two variables	**10 marks**	**Att 3**
Finish	**15 marks**	**Att 5**

1 (c) Slope of tangent = 2 $\Rightarrow$ slope of normal at $(2, 4) = -\frac{1}{2}$.

$\therefore$ Equation of normal: $(y-4) = -\frac{1}{2}(x-2) \Rightarrow 2y - 8 = -x + 2 \Rightarrow x + 2y = 10.$

Mid-point of chord joining $(2, 4)$ and $(4, -2)$ is $(3, 1)$.

Slope of chord $= \frac{4+2}{2-4} = -3.$

$\therefore$ Equation of mediatior is $y - 1 = \frac{1}{3}(x-3) \Rightarrow 3y - 3 = x - 3y = 0$.

$$\begin{array}{l} x + 2y = 10 \\ \underline{x - 3y = 0} \\ 5y = 10 \Rightarrow y = 2 \text{ and } x = 6 \Rightarrow \text{Centre is } (6, 2). \end{array}$$

$r = \sqrt{(2-6)^2 + (4-2)^2} = \sqrt{16+2} = \sqrt{20}.$

$\therefore$ Equation of circle is $(x-6)^2 + (y-2)^2 = 20.$

Or

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

(2,4) $\in$ Circle $\Rightarrow$ 20+ 4g + 8f + c =0

(4,-2) $\in$ Circle $\Rightarrow$ 20 + 8g -4f + c = 0

$\therefore g = 3f \Rightarrow$ centre (-3f, -f)

Slope of tangent = 2 $\Rightarrow$ slope of normal at $(2, 4) = -\frac{1}{2}$.

$\therefore$ Equation of normal: $(y-4) = -\frac{1}{2}(x-2) \Rightarrow 2y - 8 = -x + 2 \Rightarrow x + 2y = 10.$

. -3f +2(-f) = 10 $\Rightarrow$ f = -2 $\Rightarrow$ Centre is (6, 2)

$r = \sqrt{(2-6)^2 + (4-2)^2} = \sqrt{16+2} = \sqrt{20}$.

$\therefore$ equation of circle is $(x-6)^2 + (y-2)^2 = 20$.

First Equation:

Blunders (-3)
B1 Error in substituting into slope formula
B2 Error in substituting into midpoint formula
B3 Error in substituting into equation of line formula
B4 Incorrect signs for centre of circle

Slips (-1)
S1 Arithmetic error

Attempts (3marks)
A1 Slope of Tangent
A2 Midpoint of chord

Finish:
Slips and blunders do not apply. Award 0, 5 or 15 marks, as follows:

Fully correct: 15 marks

Attempt (5 marks)
A1 Second equation in two variables

QUESTION 2

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 15) marks**	**Att (2, 5)**
Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, -)**

Part (a) **10 marks** **Att 3**

2 (a) A, B and C are points and O is the origin.

$\vec{a} = 2\vec{i} + 3\vec{j}$, $\vec{b} = -3\vec{i} - 6\vec{j}$ and $\overrightarrow{AC} = \overrightarrow{OB}$.

Express $\vec{c}$ in terms of $\vec{i}$ and $\vec{j}$.

(a) **10 marks** **Att 3**

2 (a)

$$\overrightarrow{AC} = \overrightarrow{OB} \Rightarrow \vec{c} - \vec{a} = \vec{b} \Rightarrow \vec{c} = \vec{b} + \vec{a} = -3\vec{i} - 6\vec{j} + 2\vec{i} + 3\vec{j}.$$

$$\therefore \vec{c} = -\vec{i} - 3\vec{j}.$$

Blunders(-3)

B1 Error in $\overrightarrow{AC} = \vec{c} - \vec{a}$ or equivalent

B2 Answer not expressed in correct form

Slips (-1)

S1 Arithmetic error

Attempts (3marks)

A1 $\overrightarrow{AC} = \vec{c} - \vec{a}$ and stops

Part (b) **20 (5, 15) marks** **Att (2, 5)**

2 **(b)** $\vec{u} = 2\vec{i} + \vec{j}$ and $\vec{v} = -\vec{i} + k\vec{j}$ where $k \in \mathbb{R}$.

(i) Express $|\vec{v}|$ and $\vec{u}.\vec{v}$ in terms of k.

(ii) Given that $\cos\theta = -\frac{1}{\sqrt{2}}$, where θ is the angle between $\vec{u}$ and $\vec{v}$, find the two possible values of k.

(b) (i) **5 marks** **Att 2**

2 (b) (i)

$$|\vec{v}| = |-\vec{i} + k\vec{j}| = \sqrt{1+k^2}.$$

$$\vec{u}.\vec{v} = (2\vec{i} + \vec{j})(-\vec{i} + k\vec{j}) = -2 + k.$$

Both correct: 5 marks
One correct: 2 marks
None correct: 0 marks

(b) (ii) | **15 marks** | **Att5**

$$\cos\theta = \frac{\vec{u}.\vec{v}}{|\vec{u}|.|\vec{v}|} \Rightarrow \frac{-2+k}{\sqrt{5}\sqrt{1+k^2}} = -\frac{1}{\sqrt{2}}.$$

$$\therefore\ \sqrt{2}(-2+k) = -\sqrt{5}\sqrt{1+k^2} \Rightarrow 2(-2+k)^2 = 5+5k^2 \Rightarrow 5k^2+5 = 8-8k+2k^2$$

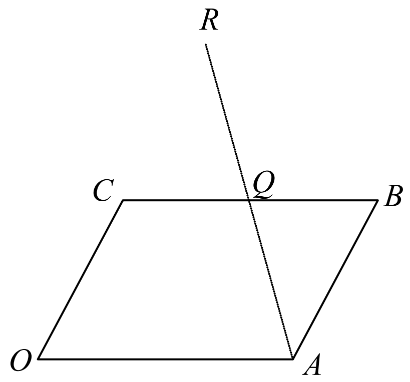
$$\therefore\ 3k^2+8k-3=0 \Rightarrow (3k-1)(k+3)=0 \Rightarrow k=\frac{1}{3}, k=-3.$$

Attempt (5 marks)

A1 Substitutes correctly

Part (c) | **20 (5, 5, 10) marks** | **Att (2, 2, -)**

2 (c) *OABC* is a parallelogram where *O* is the origin.
Q is the midpoint of $[BC]$.
$[AQ]$ is extended to *R* such that $|AQ| = |QR|$.



(i) Express $\vec{q}$ in terms of $\vec{a}$ and $\vec{c}$.

(ii) Express $\overrightarrow{AQ}$ in terms of $\vec{a}$ and $\vec{c}$.

(iii) Show that the points *O*, *C* and *R* are collinear.

(c) (i) | **5 marks** | **Att 2**

2 (c) (i)

$$\vec{q} = \vec{c} + \tfrac{1}{2}\vec{a}.$$

Blunders (-3)

B1 $\overrightarrow{CQ} \neq \frac{1}{2}\overrightarrow{OA}$

B2 Answer not in required form

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 A correct expression with $\vec{q}$

(c) (ii) **5 marks** **Att 2**

2 (c) (ii)

$$\overrightarrow{AQ} = \vec{q} - \vec{a} = \tfrac{1}{2}\vec{a} + \vec{c} - \vec{a} = \vec{c} - \tfrac{1}{2}\vec{a}.$$

Blunders (-3)

B1 $\overrightarrow{AQ} \neq \vec{q} - \vec{a}$

B2 Answer not in required form

Slips (-1)

S1 Arithmetic error

Attempts (2marks)

A1 A correct expression with $\overrightarrow{AQ}$

A2 $\overrightarrow{AQ} = \vec{q} - \vec{a}$ and stops

(c) (iii) **10 marks** **Hit /Miss**

2 (c) (iii)

$$\vec{r} = \vec{a} + \overrightarrow{AR} = \vec{a} + 2\overrightarrow{AQ} = \vec{a} + 2\vec{c} - \vec{a} = 2\vec{c}.$$

As $\vec{r} = 2\vec{c}$, then points O, C and R are collinear.

QUESTION 3

Part (a)	10 marks	Att 3
Part (b)	10 (5, 5) marks	Att (2, 2)
Part (c)	30 (10, 20) marks	Att (3, 6)

Part (a) **10 marks** **Att 3**

3 (a) The line $3x+4y-7=0$ is perpendicular to the line $ax-6y-1=0$.
Find the value of a.

(a) **10 marks** **Att 3**

3 (a)

Slope of $3x+4y-7=0$ is $-\frac{3}{4}$. Slope of $ax-6y-1=0$ is $\frac{a}{6}$.

$\therefore \frac{-3}{4}\times\frac{a}{6}=-1 \Rightarrow -3a=-24 \Rightarrow a=8.$

Blunders (-3)
B1 Error in slope
B2 Product of slopes ≠ -1
B3 Product of slopes = -1 but fails to finish

Slips (-1)
S1 Arithmetic error

Attempts (3marks)
A1 Slope of one line found

Part (b) **10(5, 5) marks** **Att (2, 2)**

3 (b) (i) The line $4x-5y+k=0$ cuts the x-axis at P and the y-axis at Q.
Write down the co-ordinates of P and Q in terms of k.

(ii) The area of the triangle OPQ is 10 square units, where O is the origin.
Find the two possible values of k.

(b) (i) **5 marks** **Att 2**

3 (b) (i)

$$P\left(\frac{-k}{4},0\right),\ Q\left(0,\frac{k}{5}\right).$$

Blunders (-3)
B1 P and Q not in coordinate form
B2 P or Q only correct

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)

A1 $\frac{-k}{4} or \frac{k}{5}$ written

A2 $\left(0, \frac{-k}{4}\right)\left(\frac{k}{5}, 0\right)$

(b) (ii) **5 marks** **Att 2**

3 (b) (ii)

$$\text{Area } \Delta OPQ = 10 \Rightarrow \frac{1}{2}\left|\left(\frac{-k}{4}\right)\left(\frac{k}{5}\right)\right| = 10. \therefore k^2 = 400 \Rightarrow k = \pm 20.$$

Blunders (-3)

B1 Error in substitution into formula for area of triangle

B2 One value of *k* only found

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Some correct substitution into formula for area of triangle

A2 k^2 = -400 or equivalent

Part (c) **30(10,20) marks** **Att (3, 6)**

3 (c) (i) *f* is the transformation $(x, y) \rightarrow (x', y')$, where $x' = x + y$ and $y' = x - y$.
The line *l* has equation $y = mx + c$.

(i) Find the equation of $f(l)$, the image of *l* under *f*.

(ii) Find the value(s) of *m* for which $f(l)$ makes an angle of 45° with *l*.

(c) (i) **10 marks** **Att 3**

3 (c) (i)

$$x' = x + y$$
$$y' = x - y$$

$$x' + y' = 2x \Rightarrow x = \frac{1}{2}(x' + y'). \quad y = x' - x = x' - \frac{1}{2}(x' + y') \Rightarrow y = \frac{1}{2}(x' - y').$$

$$f(l): \frac{1}{2}(x' - y') = \frac{m}{2}(x' + y') + c \Rightarrow x' - y' = mx' + my' + 2c.$$

$$f(l): x'(m-1) + y'(m+1) + 2c = 0.$$

Blunders (-3)

B1 Image of line not in the form $ax' + by' + c = 0$ or $y' = mx' + c$.

B2 Incorrect matrix

B3 Incorrect matrix multiplication

Slips (-1)
S1 Arithmetic error

Attempts (3marks)
A1 Expressing x or y in terms of primes
A2 Correct matrix for f when finding *f(l)*
A3 Correct image point on *f(l)*

(c) (ii) **20 marks** **Att 6**

3 (c) (ii)

Slope $l = m$ and slope $f(l) = \frac{-(m-1)}{m+1} = \frac{1-m}{1+m}$.

$$\tan 45^\circ = \left| \frac{\frac{1-m}{1+m} - m}{1 + \left(\frac{1-m}{1+m}\right)m} \right| \quad \Rightarrow \quad \left| \frac{1-m-m(1+m)}{1+m+(1-m)m} \right| = 1.$$

$$\therefore \quad \left| \frac{1-2m-m^2}{1+2m-m^2} \right| = 1 \quad \Rightarrow \quad 1-2m-m^2 = \pm\left(1+2m-m^2\right).$$

$$\therefore \quad 1-2m-m^2 = 1+2m-m^2 \quad \Rightarrow \quad 4m = 0 \quad \Rightarrow \quad m = 0.$$

OR $1-2m-m^2 = -1-2m+m^2 \quad \Rightarrow \quad -2m^2 = -2 \quad \Rightarrow \quad m^2 = 1 \quad \Rightarrow \quad m = \pm 1.$

(m = -1 gives denominator of 0 for slope of $f(l)$, but is still a solution, since in this case $f(l)$ is vertical and l makes an angle of 45° with it.)

$\therefore$ solutions are m = 0, m = 1, m = -1.

Attempt (6 marks)
A1 Substitutes correctly into formula.

Note: all three solutions not found ⇒ attempt mark at most.

QUESTION 4

Part (a)	**10 marks**	**Att 3**
Part (b)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (c)	**25 (5, 5, 15) marks**	**Att (2, 2, 5)**

Part (a) **10 marks** **Att 3**

4 (a) The area of a triangle PQR is 20 cm^2. $|PQ| = 10\text{ cm}$ and $|PR| = 8\text{ cm}$.
Find the two possible values of $|\angle QPR|$.

(a) **10 marks** **Att 3**

4 (a)

$$\text{Area } \Delta PQR = 20 \Rightarrow \tfrac{1}{2}(10)(8)\sin\angle QPR = 20.$$

$$\therefore\ \sin\angle PQR = \frac{1}{2} \ \Rightarrow\ |\angle PQR| = 30^\circ \text{ or } 150^\circ.$$

Blunders (-3)
B1 Error in substitution into area formula
B2 One angle only
B3 Angle outside the range

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Substitution into formula

Part (b)	15 (5, 5, 5) marks	Att (2, 2, 2)

4 (b) Find all the solutions of the equation $\cos 2x = \cos x$ in the domain $0^\circ \le x \le 360^\circ$.

(b) Equation	**5 marks**	**Att 2**
Roots	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

4 (b)

$$\cos 2x = \cos x \quad \Rightarrow \quad 2\cos^2 x - \cos x - 1 = 0. \quad \therefore \quad (\cos x - 1)(2\cos x + 1) = 0$$

$$\Rightarrow \quad \cos x = 1 \quad \Rightarrow \quad x = 0^\circ, x = 360^\circ \quad \text{or} \quad \cos x = -\tfrac{1}{2} \quad \Rightarrow \quad x = 120^\circ,\ 240^\circ.$$

$$\therefore \quad x = 0^\circ, 120^\circ,\ 240^\circ, 360^\circ.$$

or

4 (b)

$$\cos 2x = \cos x \;\Rightarrow\; \cos 2x - \cos x = 0 \Rightarrow -2\sin\frac{3x}{2}\sin\frac{x}{2} = 0$$

$$\Rightarrow\; \sin\frac{3x}{2} = 0 \Rightarrow\; x = 0^\circ, 120^\circ, 240^\circ, 360^\circ$$

$$\sin\frac{x}{2} = 0 \Rightarrow\; x = 0^\circ,\ 360^\circ$$

$$x = 0^\circ,\ 120^\circ,\ 240^\circ,\ 360^\circ$$

Blunders (-3)

B1 Incorrect substitution for cos $2x$

B2 Error in factors

B3 Error in substitution in quadratic formula

B4 One value omitted for either root

B5 Angle outside the domain

Slips (-1)

S1 Arithmetic error

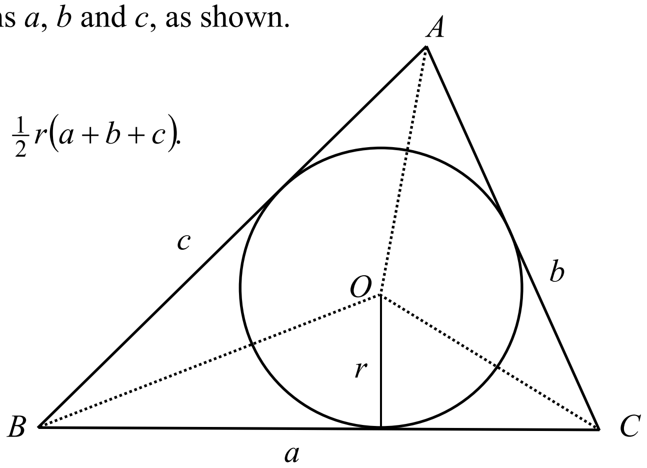
Attempts (2,2,2 marks)

A1 Cos $2x = 1 - 2\sin^2 x$ and stops

A2 Correct factors and stops

Part (c)	**25(5, 5,15) marks**	**Att (2, 2,5)**

4 (c) *ABC* is a triangle with sides of lengths *a*, *b* and *c*, as shown.
Its incircle has centre *O* and radius *r*.



(i) Show that the area of $\Delta\, ABC$ is $\frac{1}{2}r(a+b+c)$.

(ii) The lengths of the sides of a triangle are $a = p^2+q^2$, $b = p^2-q^2$ and $c = 2pq$, where *p* and *q* are natural numbers and $p > q$. Show that this triangle is right-angled.

(iii) Show that the radius of the incircle of the triangle in part **(ii)** is a whole number.

(c) (i)	**5 marks**	**Att 2**

4 (c) (i)

$$\text{Area } \Delta ABC = \tfrac{1}{2}(ar)+\tfrac{1}{2}(br)+\tfrac{1}{2}(cr)=\tfrac{1}{2}r(a+b+c).$$

Blunders (-3)
B1 Error in substitution into triangle area formula
B2 Answer not in correct format

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Area of one triangle found

(c) (ii)	**5 marks**	**Att 2**

4 (c) (ii)

$$\left(p^2+q^2\right)^2 = p^4+2p^2q^2+q^4.$$

$$\left(p^2-q^2\right)^2+\left(2pq\right)^2 = p^4-2p^2q^2+q^4+4p^2q^2 = p^4+2p^2q^2+q^4 = \left(p^2+q^2\right)^2 .$$

$\therefore$ triangle is right-angled.

Blunders (-3)
B1 Error in squaring
B2 Incorrect application of Pythagoras
B3 Conclusion not stated or implied

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)

A1 Squares any one side in terms of p and q

(c) (iii) **15 marks** **Att 5**

4 (c) (iii)

Area of $\Delta = \frac{1}{2}(2pq)(p^2 - q^2) = pq(p^2 - q^2)$.

But, by part (i),

area of $\Delta = \frac{1}{2}r(p^2 + q^2 + p^2 - q^2 + 2pq) = \frac{1}{2}r(2p^2 + 2pq) = r(p^2 + pq)$.

$$\therefore\ r(p^2 + pq) = pq(p^2 - q^2) \quad \Rightarrow \quad r = \frac{pq(p+q)(p-q)}{p(p+q)} = q(p-q).$$

As p and q are natural numbers, and $p>q$, then $p–q$ is a natural number and thus $r = q(p-q)$ is a whole number.

Attempt (5 marks)

A1 Correct expression for r in terms of p and q

QUESTION 5

Part (a)	**10 marks**	**Att 3**
Part (b)	**15 (10, 5) marks**	**Att (3, 2)**
Part (c)	**25 (10, 5, 10) marks**	**Att (3, 2, 3)**

Part (a) **10 marks** **Att 3**

5 (a) Given that $\tan\theta = \frac{1}{3}$, show that $\tan 2\theta = \frac{3}{4}$.

(a) **10 marks** **Att 3**

5 (a)

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{2}{3}\times\frac{9}{8} = \frac{3}{4}.$$

or

5 (a) $\tan\theta = \frac{1}{3} \Rightarrow \theta$ in 1st or 3rd quadrant

$\sin\theta$ and $\cos\theta$ both positive in 1st quadrant and both negative 3rd quadrant

$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{10}}$ and $\cos\theta = \pm\frac{3}{\sqrt{10}}$, (both having same sign)

In the case of both positive:

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2(\frac{1}{\sqrt{10}}).(\frac{3}{\sqrt{10}})}{(\frac{3}{\sqrt{10}})^2 - (\frac{1}{\sqrt{10}})^2} = \frac{6}{8} = \frac{3}{4}$$

and in the case of both negative:

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2(-\frac{1}{\sqrt{10}}).(-\frac{3}{\sqrt{10}})}{(-\frac{3}{\sqrt{10}})^2 - (-\frac{1}{\sqrt{10}})^2} = \frac{3}{4}$$

Blunders (-3)

B1 Error substituting into $\tan 2\theta$ formula
B2 Incorrect application of Pythagoras
B3 Error substituting into $\text{Sin } 2\theta$ and/ or $\text{Cos}2\theta$ formula(e)
B4 One quadrant only

Slips (-1)

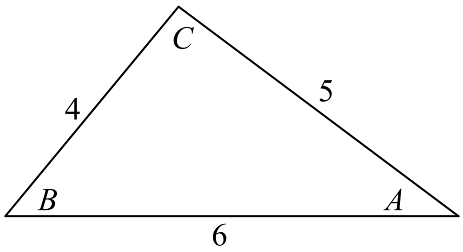
S1 Arithmetic error

Attempts (3 marks)

A1 Some substitution into $\tan 2\theta$ formula
A2 Effort at application of Pythagoras

Part (b)	**15 (10, 5) marks**	**Att (3, 2)**

5 (b) A triangle has sides of lengths 4, 5 and 6.
The angles of the triangle are A, B and C, as in diagram.



(i) Using the cosine rule, show that $\cos A + \cos C = \frac{7}{8}$.

(ii) Show that $\cos(A + C) = -\frac{9}{16}$.

(b) (i)	**10 marks**	**Att 3**

5 (b) (i)

$$\cos A = \frac{5^2 + 6^2 - 4^2}{2(5)(6)} = \frac{25 + 36 - 16}{60} = \frac{45}{60} = \frac{3}{4}$$

$$\cos C = \frac{5^2 + 4^2 - 6^2}{2(4)(5)} = \frac{25 + 16 - 36}{40} = \frac{5}{40} = \frac{1}{8}.$$

$$\therefore \quad \cos A + \cos C = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}.$$

Blunders (-3)
B1 Error substituting into cosine formula
B2 Cos A + Cos C not indicated

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Some values substituted into cosine formula for either Cos A or Cos C
A2 Cos A or Cos C formula expressed in terms of sides of a triangle
A3 Cos A or Cos C only and stops

Worthless (0)
W1 Cos A + Cos C = Cos (A+C)

(b) (ii) **5 marks** **Att 2**

5 (b) (ii)

$$\cos(A+C) = -\cos B = -\left[\frac{4^2+6^2-5^2}{2(4)(6)}\right] = -\left[\frac{16+36-25}{48}\right] = -\left[\frac{27}{48}\right] = -\frac{9}{16}.$$

or

5 (b) (ii)

$$\text{Cos }(A+C) = \text{ Cos } A \text{ Cos } C - \text{Sin}A \text{ Sin } C = \frac{3}{4}.\frac{1}{8} - \frac{\sqrt{7}}{4}.\frac{3\sqrt{7}}{8} = \frac{-18}{32} = \frac{-9}{16}$$

Blunders (-3)

B1 Cos (A+C) ≠ - Cos B

B2 Incorrect ratio for Sin A or Sin C

B3 Error substituting into expansion of Cos (A + C)

B4 Conclusion not stated or implied

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Cos (A + C) = Cos ($180^0 - B$) and stops

A2 Some substitution into Cos (A+C) expansion

A3 Use of Pythagoras

Worthless (0)

W1 Cos (A + C) = Cos A+Cos$C$

W2 Cos (A + C) = Cos A. Cos C

Part (c) **25(10, 5, 10) marks** **Att (3, 2, 3)**

5 (c) (i) Show that $(\cos A+\cos B)^2+(\sin A+\sin B)^2 = 2+2\cos(A-B)$.

(ii) Hence solve the equation $(\cos 4x+\cos x)^2+(\sin 4x+\sin x)^2 = 2+2\sqrt{3}\sin 3x$ in the domain $0^\circ \le x \le 360^\circ$.

(c) (i) **10 marks** **Att 3**

5 (c) (i)

$$(\cos A+\cos B)^2+(\sin A+\sin B)^2$$
$$= \cos^2 A+2\cos A\cos B+\cos^2 B+\sin^2 A+2\sin A\sin B+\sin^2 B$$
$$= (\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)+2(\cos A\cos B+\sin A\sin B)$$
$$= 2+2\cos(A-B).$$

Blunders (-3)

B1 Error in squaring

B2 $\cos^2 A+\sin^2 A \ne 1$

B3 cosAcosB + sinAsinB ≠ cos ($A - B$)

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 $(\cos A + \cos B)^2$ or equivalent correct

(c) (ii) tan 3*x* **5 marks** **Att 2**
Solutions **10 marks** **Att 3**

5 (c) (ii)

$$(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2 + 2\cos 3x \quad \text{by part (i).}$$

$$\therefore\ 2 + 2\cos 3x = 2 + 2\sqrt{3}\sin 3x \ \Rightarrow\ \sqrt{3}\sin 3x = \cos 3x \ \Rightarrow\ \frac{\sin 3x}{\cos 3x} = \frac{1}{\sqrt{3}}.$$

$$\therefore\ \tan 3x = \frac{1}{\sqrt{3}} \ \Rightarrow\ 3x = 30^\circ, 210^\circ, 390^\circ, 570^\circ, 750^\circ, 930^\circ.$$

$$\therefore\ x = 10^\circ, 70^\circ, 130^\circ, 190^\circ, 250^\circ, 310^\circ.$$

Or

(c) (ii) 2cos (3x + 60 ^{0}) **5 marks** **Att 2**
Solutions **10 marks** **Att 3**

$$(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2 + 2\cos 3x, \quad \text{by part (i).}$$

$$\therefore\ 2 + 2\cos 3x = 2 + 2\sqrt{3}\sin 3x \ \Rightarrow\ \sqrt{3}\sin 3x = \cos 3x \ \Rightarrow \sqrt{3}\ \sin 3x - \cos 3x = 0$$

$$\Rightarrow\ \cos 3x - \sqrt{3}\sin 3x = 0 \Rightarrow 2\left(\frac{1}{2}\cos 3x - \frac{\sqrt{3}}{2}\sin 3x\right) = 0$$

$$\Rightarrow\ 2\cos(3x + 60^\circ) = 0 \Rightarrow 3x + 60^\circ = 90^\circ, 270^\circ, 450^\circ, 630^\circ, 810^\circ, 990^\circ$$

$$\therefore\ x = 10^\circ, 70^\circ, 130^\circ, 190^\circ, 250^\circ, 310^\circ.$$

Equation:
Attempts (2 marks)
A1 A correct manipulation

Worthless (0)
W1 Cos 4*x* = 4 Cos *x* or equivalent

Solutions:
Attempt (3marks)
A1 Solution set with one omitted or incorrect value

Worthless (0)
W1 Solution set with more than one omitted or incorrect value

QUESTION 6

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 marks**	**Att 6**

Part (a) 10 marks Att 3

6 (a) One bag contains four red discs and six blue discs.
Another bag contains five red discs and seven yellow discs.
One disc is drawn from each bag.
What is the probability that both discs are red?

(a) 10 marks Att 3

6 (a)

Numbers of favourable outcomes $= {}^{4}C_1 \times {}^{5}C_1 = 20.$

Numbers of possible outcomes ${}^{10}C_1 \times {}^{12}C_1 = 120.$

$\therefore$ Probability both discs are red $= \dfrac{20}{120} = \dfrac{1}{6}.$

Or

$\text{P(Both red discs)} = \dfrac{4}{10} \times \dfrac{5}{12} = \dfrac{1}{6}.$

Blunders (-3)

B1 Incorrect number of possible outcomes

B2 Answer not in form of $\dfrac{a}{b}, \quad a \in \mathbb{N}, b \in \mathbb{N}$

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Correct number of possible outcomes

A2 Correct number of favourable outcomes

A3 ${}^{4}C_1 + {}^{5}C_1$ or equivalent, with or without further work

Part (b)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

6 (b) α and β are the roots of the quadratic equation $px^2 + qx + r = 0$.

$u_n = l\alpha^n + m\beta^n$, for all $n \in \mathbb{N}$.

Prove that $pu_{n+2} + qu_{n+1} + ru_n = 0$, for all $n \in \mathbb{N}$

(b) Uses root property correctly	**5 marks**	**Att 2**
Deduces u_{n+1}, u_{n+2}	**5 marks**	**Att 2**
Substitutes and tidies up	**5 marks**	**Att 2**
Conclusion	**5 marks**	**Att 2**

6 (b)

α is a root of $px^2 + qx + r = 0 \Rightarrow p\alpha^2 + q\alpha + r = 0$

Similarly: $p\beta^2 + q\beta + r = 0$

Given: $u_n = l\alpha^n + m\beta^n \Rightarrow u_{n+1} = l\alpha^{n+1} + m\beta^{n+1}, u_{n+2} = l\alpha^{n+2} + m\beta^{n+2}$

$$\Rightarrow pu_{n+2} + qu_{n+1} + ru_n = p(l\alpha^{n+2} + m\beta^{n+2}) + q(l\alpha^{n+1} + m\beta^{n+1}) + r(l\alpha^n + m\beta^n)$$
$$= l\alpha^n(p\alpha^2 + q\alpha + r) + m\beta^n(p\beta^2 + q\beta + r)$$
$$= l\alpha^n(0) + m\beta^n(0)$$
$$= 0$$

Blunders (-3)

B1 Fails to use root property correctly
B2 Error in expressing value of term
B3 Error in substituting or tidying
B4 Incorrect conclusion or no conclusion implied

Slips (-1)

S1 Arithmetic error

Attempts (2,2,2,2 marks)

A1 Effort at substituting either root into quadratic
A2 Some correct substitution for u_{n+1} or equivalent

Part (c) **20 marks** **Att 6**

6 (c) In a café there are 11 seats in a row at the counter.
Six people are seated at the counter. How much more likely is it that all six are seated together than that no two of them are seated together?

(c) **20 marks** **Att 6**

6 (c) *Taking arrangements as unordered:*

Number of possible ways of seating six people in a row of 11 seats $= {}^{11}C_6 = 462$.

To seat six people together, seat them in seats 1 to 6, or 2 to 7, or 3 to 8, or 4 to 9, or 5 to 10, or 6 to 11.

$\therefore$ Number of favourable outcomes = 6.

$\therefore$ Probability of six seated together $= \dfrac{6}{462}$.

In order to seat six people with no two together, seat them in seat 1, seat 3, seat 5, seat 7, seat 9 and seat 11. There is no other possible way to seat them. There is only one favourable outcome.

$\therefore$ Probability of no two of them seated together $= \dfrac{1}{462}$.

$\therefore$ It is six times more likely that all six people are seated together.

OR

6 (c) *Taking arrangements as ordered:*

Number of possible ways of seating six people in a row of 11 seats $= {}^{11}P_6 = 332640$.

To seat six people together, seat them in seats 1 to 6, or 2 to 7, or 3 to 8, or 4 to 9, or 5 to 10, or 6 to 11.

$\therefore$ Number of favourable outcomes = $6 \times 6! = 4320$.

$\therefore$ Probability of six seated together $= \dfrac{4320}{332640}$.

In order to seat six people with no two together, seat them in seat 1, seat 3, seat 5, seat 7, seat 9 and seat 11. There is no other possible way to seat them. So there are $1 \times 6! = 720$ favourable outcomes.

$\therefore$ Probability that no two of them seated together $= \dfrac{720}{332640}$.

4320 = 6 × 720, so it is six times more likely that all six people are seated together.

Attempt (6marks)

A1 Correct expression for one or other case

* Note special case: If the candidate has both probabilities correct but subtracts them instead of dividing: award17 marks

* Apart from the special case mentioned, award 0, 6, or 20 marks.

QUESTION 7

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (10, 10) marks**	**Att (3, 3)**

Part (a) **10 marks** **Att 3**

7 (a) A password for a website consists of capital letters A, B, C,. … Z and/or digits 0, 1, 2, …. 9.
The password has four such characters and starts with a letter. For example, BA7A, C999 and DGKK are allowed, but 7DCA is not.
Show that there are more than a million possible passwords.

(a) **10 marks** **Att 3**

7 (a) Numbers of possible passwords $= 26 \times 36 \times 36 \times 36 = 1{,}213{,}056 > 1{,}000{,}000$.

or

7 (a) Numbers of possible passwords:
26.26.26.26 + 3(26.26.26.10) + 3(26.26.10.10) + 26.10.10.10 = 1,213,056 >1,000,000.

Attempt (3marks)
A1 Solution with one error

Part (b) **20(5, 5, 10) marks** **Att (2, 2, 3)**

7 (b) Karen is about to sit an examination at the end of an English course.
The course has twenty prescribed texts.
Six of these are novels, four are plays and ten are poems.
The examination consists of a question on one of the novels, a question on one of the plays and a question on one of the poems.
Karen has studied four of the novels, three of the plays and seven of the poems.

Find the probability that:

(i) Karen has studied all three of the texts on the examination

(ii) Karen has studied none of the texts on the examination

(iii) Karen has studied at least two of the texts on the examination.

(b) (i) **5 marks** **Att 2**

7 (b) (i)

$$\text{Probability (studies all three texts)} = \frac{4 \times 3 \times 7}{6 \times 4 \times 10} = \frac{84}{240} = \frac{7}{20}.$$

Blunders (-3)
B1 Incorrect number of possible outcomes

B2 Answer not expressed in form of $\frac{a}{b}, a \in \mathbb{N}, b \in \mathbb{N}$ or equivalent

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Correct number of possible outcomes
A2 Correct number of favourable outcomes
A3 $\frac{4}{6}+\frac{3}{4}+\frac{7}{10}$ with or without further work

(b) (ii) **5 marks** **Att 2**

7 (b) (ii)

Probability (studies none of the texts) $= \frac{2\times1\times3}{6\times4\times10} = \frac{6}{240} = \frac{1}{40}$.

Blunders (-3)
B1 Incorrect number of possible outcomes
B2 Answer not expressed in form of $\frac{a}{b}, a\in\mathbb{N}, b\in\mathbb{N}$ or equivalent

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Correct number of possible outcomes
A2 Correct number of favourable outcomes
A3 $\frac{2}{6}+\frac{1}{4}+\frac{3}{10}$ with or without further work

(b) (iii) **10 marks** **Att 3**

7 (b) (iii)

Probability (studies at least two of the texts)
=Probability (studies two) + Probability (studies three)

$$=\left(\frac{4\times3\times3}{6\times4\times10}+\frac{4\times1\times7}{6\times4\times10}+\frac{2\times3\times7}{6\times4\times10}\right)+\frac{84}{240}=\frac{36+28+42}{240}+\frac{84}{240}=\frac{190}{240}=\frac{19}{24}.$$

Attempt (3marks)
A1 Correct expression all three terms in P(2 texts) or all three terms in P(1 text).

Part (c)	**20(10, 10) marks**	**Att (3, 3)**

(c) The mean of the real numbers a, $2a$, $3a$, $4a$ and $5a$ is μ and the standard deviation is σ.

(i) Express μ and σ in terms of a.

(ii) Hence write down in terms of a, the mean and the standard deviation of $3a+5,\ 6a+5,\ 9a+5,\ 12a+5,\ 15a+5$.

(c) (i)	**10 marks**	**Att 3**

7 (c) (i)

$$\mu = \frac{a+2a+3a+4a+5a}{5} = \frac{15a}{5} = 3a.$$

$$\sigma^2 = \frac{(a-3a)^2+(2a-3a)^2+(3a-3a)^2+(4a-3a)^2+(5a-3a)^2}{5} = \frac{4a^2+a^2+a^2+4a^2}{5}$$

$$= \frac{10a^2}{5} = 2a^2.$$

$$\therefore\ \sigma = \sqrt{2}a.$$

Attempt (3marks)

A1 Expression for mean or standard deviation correct.

(c) (ii)	**10 marks**	**Att3**

7 (c) (ii)

Mean = $3\mu + 5 = 9a + 5$.

Standard deviation = $3\sigma = 3\sqrt{2}a$.

* If not ‘Hence’ (i.e. otherwise) 3 marks for mean and/or standard deviation correct

Attempt (3marks)

A1 Mean or standard deviation correct

QUESTION 8

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 15) marks**	**Att (- , 5)**
Part (c)	**20 (10, 5, 5) marks**	**Att (3, - , -)**

Part (a) **10 marks** **Att 3**

8 (a) Use integration by parts to find $\int \log_e x\, dx$.

(a) **10 marks** **Att 3**

8 (a)

$$\int u dv = uv - \int v du. \quad \text{Let } u = \log_e x \Rightarrow du = \frac{1}{x}dx. \quad dv = dx \Rightarrow v = x.$$

$$\therefore \int \log_e x dx = x\log_e x - \int x\left(\frac{1}{x}\right)dx = x\log_e x - \int dx = x\log_e x - x + C.$$

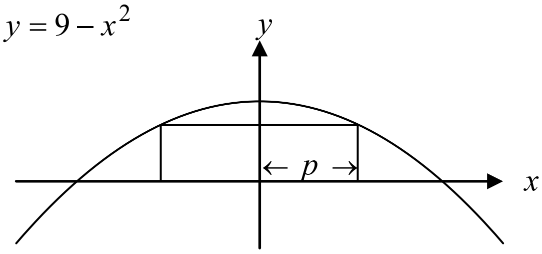
Blunders (-3)
B1 Incorrect differentiation or integration
B2 Incorrect 'parts' formula

Slips (-1)
S1 Arithmetic error
S2 Omits constant of integration

Attempts (3 marks)
A1 One correct assigning to parts formula
A2 Correct differentiation or integration

Part (b) **20(5, 15) marks** **Att (- ,5)**

8 (b) A rectangle is inscribed between the curve $y = 9 - x^2$ and the x-axis, as shown.



(i) Write an expression for the area of the rectangle in terms of p.

(ii) Hence, calculate the area of the largest possible rectangle.

(b) (i) **5 marks** **Hit/Miss**

8 (b) (i)

Length of rectangle $= 2p$ and its width $= 9 - p^2$.

Area of rectangle $= A = 2p\left(9 - p^2\right) = 18p - 2p^3$.

(b) (ii) **15 marks** **Att 5**

8 (b) (ii)

$$\therefore \frac{dA}{dp} = 18 - 6p^2. \text{ For maximum, } \frac{dA}{dp} = 0 \Rightarrow 18 - 6p^2 = 0 \Rightarrow p = \sqrt{3}.$$

$$\frac{d^2A}{dp^2} = -12p < 0 \text{ for } p = \sqrt{3}.$$

$$\therefore A = 18\sqrt{3} - 6\sqrt{3} = 12\sqrt{3} \text{ is largest possible rectangle.}$$

* Note: If candidate gets no marks for (b)(i), then cannot get any marks for (b)(ii).

Attempt (5 marks)

A1 Correct differentiation

Part (c) **20 (10, 5, 5) marks** **Att (3, - , -)**

8 (c) (i) Derive the Maclaurin series for $f(x) = \cos x$, up to and including the term containing x^6.

(ii) Hence, and using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, show that the first three non zero terms of the Maclaurin series for $\sin^2 x$ are $x^2 - \frac{x^4}{3} + \frac{2x^6}{45}$.

(iii) Use these terms to find an approximation for $\text{Sin}^2\left(\frac{1}{2}\right)$, as a fraction.

(c) (i) **10 marks** **Att 3**

8 (c) (i)

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \ ....$$

$$f(x) = \cos x \quad \Rightarrow \quad f(0) = \cos 0 = 1.$$

$$f'(x) = -\sin x \quad \Rightarrow \quad f'(0) = -\sin 0 = 0.$$

$$f''(x) = -\cos x \quad \Rightarrow \quad f''(0) = -\cos 0 = -1.$$

$$f'''(x) = \sin x \quad \Rightarrow \quad f'''(0) = \sin 0 = 0.$$

$$f^{(4)}(x) = \cos x \quad \Rightarrow \quad f^{(4)}(0) = \cos 0 = 1.$$

$$f^{(5)}(x) = -\sin x \quad \Rightarrow \quad f^{(5)}(0) = -\sin 0 = 0.$$

$$f^{(6)}(x) = -\cos x \quad \Rightarrow \quad f^{(6)}(0) = -\cos 0 = -1.$$

$$\therefore \ f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Blunders (-3)

B1 Incorrect differentiation

B2 Incorrect evaluation of $f^{(n)}(0)$

B3 Each term not derived (to max of 2)

B4 Error in Maclaurin series

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Correct expansion for Cos$x$ given but not derived

A2 $f(0)$ correct

A3 A correct differentiation

A4 Any one correct term

(c) (ii) **5 marks** **Hit/Miss**

8 (c) (ii)

By part (i), $\cos 2x = 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \frac{64x^6}{720} = 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45}$.

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) = \tfrac{1}{2}\left(1 - \left\{1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45}\right\}\right) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45}.$$

(c) (iii) **5 marks** **Hit/Miss**

8 (c) (iii)

$$\sin^2\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2 - \frac{\left(\tfrac{1}{2}\right)^4}{3} + \frac{2\left(\tfrac{1}{2}\right)^6}{45} = \frac{1}{4} - \frac{1}{48} + \frac{1}{1440} = \frac{360 - 30 + 1}{1440} = \frac{331}{1440}.$$

QUESTION 9

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (5,5, 5, 5) marks**	**Att (2,2, 2, 2)**

Part (a) **10 marks** **Att 3**

9 (a) Z is a random variable with standard normal distribution.
Find $P(-1 < Z \leq 1)$.

Part (a) **10 marks** **Att 3**

9 (a)

$$P(-1 < Z \leq 1) = P(Z \leq 1) - [1 - P(Z \leq 1)] = 2(0{\cdot}8413) - 1 = 0{\cdot}6826$$

or

9 (a)

$$P(-1 < Z \leq 1) = 2\,(P(Z \leq 1) - P(Z \leq 0)) = 2(\,0{\cdot}8413 - 0{\cdot}5) = 0{\cdot}6826$$

Blunders (–3)

B1 $P(z \leq 1)$ incorrect or $P(Z \leq 0)$ incorrect

B2 Mishandles $P(-1 < Z)$

Slips (–1)

S1 Arithmetic error.

Attempts (3 marks)

A1 $P(Z \leq 1)$ correct.

Part (b) **20 (5, 5, 10) marks** **Att (2, 2, 3)**

9 (b) A test consists of twenty multiple-choice questions. Each question has four possible answers, only one of which is correct. Seán decides to guess all the answers at random. Find the probability that:

(i) Seán gets none of the answers correct

(ii) Seán gets exactly five of the answers correct

(iii) Seán gets four, five or six of the answers correct.
Give each of your answers correct to three decimals places.

(b) (i) **5 marks** **Att 2**

9 (b) (i)

$$p = \tfrac{1}{4},\ q = \tfrac{3}{4}.$$

$$\text{Probability (none correct)} = \left(\frac{3}{4}\right)^{20} \approx 0{\cdot}003\,.$$

Blunders (-3)
B1 Incorrect p or q
B2 Binomial error
B3 Answer not in required form

Slips (-1)
S1 Arithmetic error
S2 Answer not to two decimal places

Attempts (2 marks)
A1 Correct p or q

(b) (ii) **5 marks** **Att 2**

9 (b) (ii)

$$\text{Probability (exactly five correct)} = {}^{20}C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{15} = 0{\cdot}202\,.$$

Blunders (-3)
B1 Binomial error
B2 Answer not in decimal form

Slips (-1)
S1 Arithmetic error
S2 Answer not to 3 decimal places

Attempts (2 marks)
A1 ${}^{20}C_5$ used or implied

(b) (iii) **10 marks** **Att 3**

9 (b) (iii)

Probability (four, five or six) =

$${}^{20}C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{16} + {}^{20}C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^{15} + {}^{20}C_6 \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^{14}$$

$$= 0{\cdot}1896 + 0{\cdot}2023 + 0{\cdot}1686 = 0{\cdot}5605 = 0{\cdot}561$$

Blunders (-3)
B1 Each term omitted
B2 Binomial error
B3 Answer not in decimal form
B4 Rounding off too early

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Effort at probability of four or six correct

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

9 (c) A bakery produces muffins. A random sample of 50 muffins is selected and weighed. The mean of the sample is 80 grams and the standard deviation is 6 grams.
Form a 95% confidence interval for the mean weight of muffins produced by the bakery.

(c) Correct S.E.	**5 marks**	**Att 2**
Two tailed	**5 marks**	**Att 2**
Mean ± 1.96 SE	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

9 (c)

$\bar{x} = 80,\ \sigma = 6$ and $n = 50$.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{6}{\sqrt{50}} = \frac{6}{5\sqrt{2}} = \frac{3\sqrt{2}}{5}.$$

The 95% confidence interval is

$$\left[\bar{x} - 1{\cdot}96(\sigma_{\bar{x}}), \bar{x} + 1{\cdot}96(\sigma_{\bar{x}})\right]$$

$$= \left[80 - 1{\cdot}96\left(\frac{3\sqrt{2}}{5}\right),\ 80 + 1{\cdot}96\left(\frac{3\sqrt{2}}{5}\right)\right] = [78{\cdot}3,\ 81{\cdot}6] \text{ grams.}$$

Blunders (–3)

B1 Error in standard error of mean.
B2 Error from tables.
B3 Answer not simplified.

Slips (–1)

S1 Arithmetic error.

Attempts (2,2, 2, 2 marks)

A1 Standard error of mean with some substitution.
A2 Incomplete substitution.

QUESTION 10

Part (a)	**10(5, 5) marks**	**Att (2, 2)**
Part (b)	**40(5,5,5,5,5,5,5,5) marks**	**Att (2,2,2,2,2,2,2,2)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

10 (a) The binary operation $*$ is defined by $x * y = x + y - xy$, where $x, y \in \mathbb{R} \setminus \{-1\}$.

(i) Find the identity element.

(ii) Express x^{-1}, the inverse of x, in terms of x.

(a) (i) **5 marks** **Att 2**

10 (a) (i)

$$x * e = x + e - xe = x \quad \Rightarrow \quad e(1-x) = 0 \quad \forall x \Rightarrow \quad e = 0.$$

Blunders (-3)

B1 $x * e$ incorrect

B2 $e - xe = 0$ and stops

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 $x * e = x$

A2 $x * e = x + e - xe$

(a) (ii) **5 marks** **Att 2**

10 (a) (ii)

$$x * x^{-1} = e \quad \Rightarrow \quad x + x^{-1} - xx^{-1} = 0.$$

$$\therefore x^{-1}(1-x) = -x \quad \Rightarrow \quad x^{-1} = \frac{x}{x-1}, \text{ (provided } x \neq 1\text{).}$$

Blunders (-3)

B1 $x * x^{-1}$ incorrect

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 $x * x^{-1}$ correct and stops

A2 $x * x^{-1} = 0$

Part (b) **40(5, 5, 5, 5, 5, 5, 5, 5) marks** **Att (2, 2, 2, 2, 2, 2, 2, 2)**

10 (b) G is the set of permutations of $\{1, 2, 3\}$ and the six elements of G are as follows:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

$(G, \circ)$ is a group, where $\circ$ denotes composition.

(i) Write down b^{-1} and d^{-1}, the inverses of b and d respectively.

(ii) Verify that $(b \circ d)^{-1} = d^{-1} \circ b^{-1}$.

(iii) Write down the subgroups of $(G, \circ)$ of order 2.

(iv) K is the subgroup of $(G, \circ)$ of order 3. List the elements of K.

(v) $(H, \times)$ is a group, where $H = \{1, w, w^2\}$ and $w^3 = 1$.
Give an isomorphism ϕ from $(K, \circ)$ to $(H, \times)$, justifying fully that it is an isomorphism.

(b) (i) $\boldsymbol{b^{-1}}$ **5 marks** **Att 2**
$\boldsymbol{d^{-1}}$ **5 marks** **Att 2**

10 (b) (i)

$$b^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad d^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Blunders (-3)
B1 Incorrect element (max of 2)

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Permutation incomplete
A2 One element correct with another repeated

(b) (ii) One composition correct **5 marks** **Att 2**
Finish **5 marks** **Att 2**

10 (b) (ii)

$$(b \circ d)^{-1} = \left[\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right]^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

$$d^{-1} \circ b^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (b \circ d)^{-1}.$$

Blunders (-3)
B1 Incorrect element (max 2)
B2 $d \circ b$ 'correct' instead of $b \circ d$
B2 Incorrect conclusion or no conclusion implied

Slips (-1)
S1 Arithmetic error

Attempts (2,2 marks)
A1 Permutation incomplete
A2 One element correct with another repeated

(b) (iii)	**5 marks**	**Att 2**
10 (b) (iii)	$\{a, b\}$, $\{a, d\}$, $\{a, g\}$are subgroups of order two.	

Blunders (-3)
B1 Subgroup omitted
B2 Incorrect subgroup

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 One correct subgroup

(b) (iv)	**5 marks**	**Att 2**
10 (b) (iv)	$K = \{a, c, f\}$ is a subgroup of order three.	

Blunders (-3)
B1 One incorrect element (other than identity)

Slips (-1)
S1 Arithmetic error

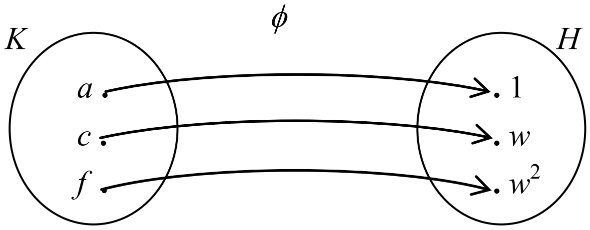
Attempts (2 marks)
A1 {a,b,d}

Worthless (0)
W1 $\{ b,d,g\}$
W2 No identity element

(b) (v) Establishing link **5marks** **Att 2**

Finish **5 marks** **Att 2**

10 (b) (v)



a and 1 are the identities of $(K, \circ)$ and $(H, \times)$ respectively.

$\phi(c \circ c) = \phi(f) = w^2$ and $\phi(c) \times \phi(c) = w \times w = w^2$.

$\phi(f \circ f) = \phi(c) = w$ and $\phi(f) \times \phi(f) = w^2 \times w^2 = w^4 = w$.

$\phi(c \circ f) = \phi(a) = 1$ and $\phi(c) \times \phi(f) = w \times w^2 = w^3 = 1$.

$\phi(f \circ c) = \phi(a) = 1$ and $\phi(f) \times \phi(c) = w^2 \times w = w^3 = 1$.

$\phi(a \circ a) = \phi(a) = 1$ and $\phi(a) \times \phi(a) = 1 \times 1 = 1$

$\phi(a \circ c) = \phi(c) = w$ *and* $\phi(a) \times \phi(c) = w$

$\phi(a \circ f) = \phi(f) = w^2$ *and* $\phi(a) \times \phi(f) = w^2$

$\phi(c \circ a) = \phi(c) = w$ *and* $\phi(c) \times \phi(a) = w$

$\phi(f \circ a) = \phi(f) = w^2$ *and* $\phi(f) \times \phi(a) = w^2$

$\therefore$ Isomorphism.

Alternative Methods:

K is a cyclic group with generator *c*.

K: *{a,f,c}* $\rightarrow$ $\{c^3, c, c^2\}$

H is a cyclic group with generator *w*

Isomorphism: $c^3 \leftrightarrow 1(or\, w^3), c \leftrightarrow w, c^2 \leftrightarrow w^2$

<u>Justification</u>:

K and *H* are both cyclic groups of same order (order 3)

$\Rightarrow$ *K* and *H* isomorphic, under any function that maps a generator to a generator and corresponding powers accordingly, as this one does.

or (alternative justification)

Theorem: Any cyclic group of order *n* is isomorphic to the group of complex *n*th roots of unity

K is s cyclic group of order 3, and *H* is the group of the cubic roots of unity $\Rightarrow$ *K* and *H* isomorphic under this function.

* Using alternative methods above, it is not sufficient to show the groups are isomorphic; an isomorphism must also be given.

Blunders (-3)

B1 Cayley table but links not established

B2 Incomplete justification

Slips (-1)

S1 Arithmetic error

Attempts (2,2 marks)

A1 Link identities only

A2 States order of groups.

QUESTION 11

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (10, 10) marks**	**Att (3, 3)**

Part (a) **10 marks** **Att 3**

11 (a) An ellipse with centre $(0, 0)$ has eccentricity $\frac{4}{5}$ and the length of its major axis is 2 units. Find its equation.

(a) **10 marks** **Att 3**

11 (a)

$$2a = 2 \Rightarrow a = 1. \quad b^2 = a^2\left(1 - e^2\right) \quad \Rightarrow \quad b^2 = 1\left(1 - \frac{16}{25}\right) \Rightarrow \quad b^2 = \frac{9}{25}.$$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad x^2 + \frac{25y^2}{9} = 1.$

Blunders (-3)
B1 Incorrect a
B2 b^2 calculated, but equation not found
B3 Error in forming equation

Slips (-1)
S1 Arithmetic error

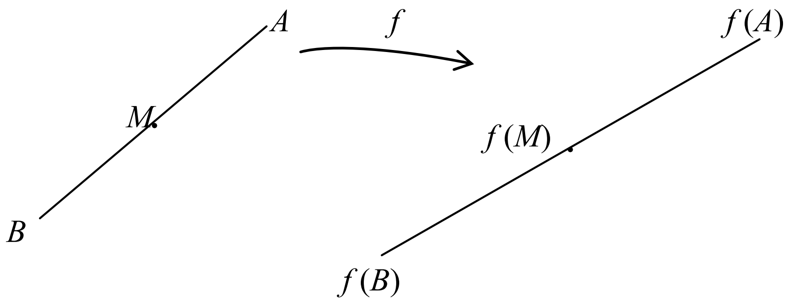
Attempts (3 marks)
A1 $a = 1$
A2 Some substitution into b^2 formula

Part (b) **20(10, 10) marks** **Att (3, 3)**

11 (b) f is an affine transformation. The point M is the mid-point of the line segment $[AB]$.
(i) Show that $f(M)$ is the mid-point of the line segment $[f(A)f(B)]$.
(ii) A triangle ABC has centroid G.
Show that the triangle $f(A)f(B)f(C)$ has centroid $f(G)$.

(c)(i) **10 marks** **Att 3**

11 (b) (ii)



M is on $AB \Rightarrow f(M)$ is on $f(AB)$.
M is mid-point of $[AB] \;\Rightarrow\; |AM|:|MB| = 1:1$.
Ratio of lengths on parallel lines is an affine invariant.
But AM is parallel to $MB \quad \Rightarrow \dfrac{|f(A)f(M)|}{|f(M)f(B)|} = \dfrac{|AM|}{|MB|} = \dfrac{1}{1}$

$$\Rightarrow \; f(M) \text{ is mid-point of } [f(A)f(B)].$$

or

Let f be the affine transformation such that $(x,y) \to (x', y')$ so that

$x' = ax + by + k \qquad y' = cx + dy + h, \; a, b, c, d, k, h \in \mathbb{R}$ and $ad - bc \neq 0$.

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of A and B.

$f(A) = (ax_1 + by_1 + k, cx_1 + dy_1 + h)$ and $f(B) = (ax_2 + by_2 + k, cx_2 + dy_2 + h)$

Midpoint of $[f(A)f(B)] = \left(\dfrac{a(x_1+x_2) + b(y_1+y_2) + 2k}{2}, \dfrac{c(x_1+x_2) + d(y_1+y_2) + 2h}{2} \right)$

But M, (the midpoint of $[AB]$), is $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2} \right)$.

$$\therefore f(M) = \left(a\left(\frac{x_1+x_2}{2}\right) + b\left(\frac{y_1+y_2}{2}\right) + k, \;\; c\left(\frac{x_1+x_2}{2}\right) + d\left(\frac{y_1+y_2}{2}\right) + h \right)$$
$$= \left(\frac{a(x_1+x_2) + b(y_1+y_2) + 2k}{2}, \frac{c(x_1+x_2) + d(y_1+y_2) + 2h}{2} \right)$$
$$= \text{midpoint of } [f(A)f(B)].$$

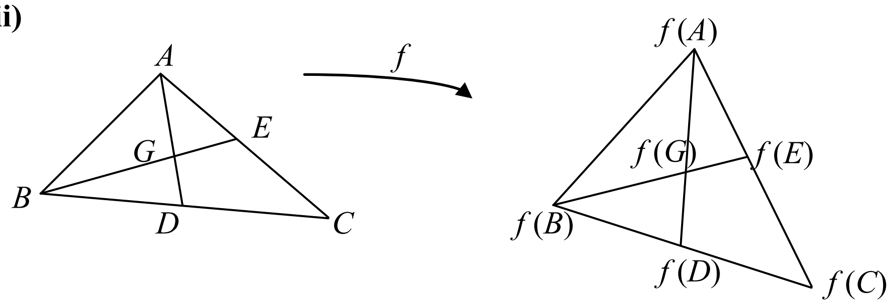
Blunders (-3)
B1 Fails to establish relationship between M and end points of segment A and B
B2 Fails to establish relationship between segment length and its image under f
B3 Incorrect conclusion

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Shows some relevant mapping

(b) (ii) **10 marks** **Att 3**

11 (b) (ii)



D and E are mid-points of $[BC]$ and $[AC]$ respectively $\Rightarrow G$ is centroid of ΔABC.

$[AD]$ and $[BE]$, under f, map to $[f(A)f(D)]$ and $[f(B)f(E)]$ respectively.

But mid-point is an affine invariant,
$\Rightarrow f(D)$ and $f(E)$ are the mid-points of $[f(B)f(C)]$ and $[f(A)f(C)]$ respectively.
$\therefore$ $[f(A)f(D)] \cap [f(\text{B})f(E)] = f(G)$ is the centroid of $\Delta f(A)f(B)f(C)$.

Blunders (-3)
B1 Fails to define centroid
B2 Fails to state mid point invariant
B3 Fails to state that f(G) centroid

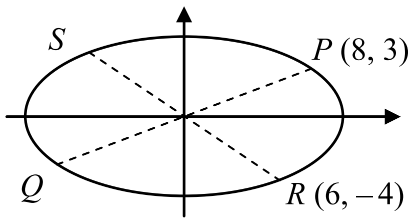
Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Shows some relevant mapping

Part (c) **20(10, 10) marks** **Att (3, 3)**

11 (c) An ellipse e has equation $\frac{x^2}{100}+\frac{y^2}{25}=1$.

$[PQ]$ and $[RS]$ are diameters of the ellipse, where P is $(8,3)$ and R is $(6,-4)$.



(i) Using a transformation to or from the unit circle, or otherwise, show that the diameters $[PQ]$ and $[RS]$ are conjugate.

(ii) Find the area of the parallelogram that circumscribes the ellipse at the points P, S, Q, and R.

(c)(i) **10 marks** **Att 3**

11 (c) (i)

f is the transformation $(x, y)\rightarrow(x', y')$ where $x'=\frac{x}{10}$, $y'=\frac{y}{10}$.

Therefore, $x=10x'$, $y=5y'$

$\therefore\ f(e):\frac{100x'^2}{100}+\frac{25y'^2}{25}=1\Rightarrow x'^2+y'^2=1$.

Also $f(P)=\left(\frac{8}{10},\frac{3}{5}\right)=\left(\frac{4}{5},\frac{3}{5}\right)$, $f(R)=\left(\frac{6}{10},\frac{-4}{5}\right)=\left(\frac{3}{5},\frac{-4}{5}\right)$. Also $f(0,0)=(0,0)$.

Slope $f(P)f(Q)=\frac{\frac{3}{5}-0}{\frac{4}{5}-0}=\frac{3}{4}$ and slope $f(R)f(S)=\frac{\frac{-4}{5}-0}{\frac{3}{5}-0}=\frac{-4}{3}$.

But $\frac{3}{4}\times\frac{-4}{3}=-1\Rightarrow[f(P)f(Q)]$ and $[f(R)f(S)]$ are conjugate diameters in the circle.

$\therefore$ diameters $[PQ]$ and $[RS]$ are conjugate diameters in the ellipse.

Blunders (-3)
B1 Error in image of co-ordinates under transformation
B2 Error in substitution into slope formula
B3 Conclusion not justified or incorrect conclusion

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Image of one point correct
A2 x^1 or equivalent correct

(c) (ii) **10 marks** **Att 3**

11 (c) (ii)
The area of the square that circumscribes the circle at the points $f(P), f(S), f(Q), f(R)$ is $4r^2 = 4$ square units.
Area of parallelogram $PSQR = \left\|\det f^{-1}\right\|$.(Area of square $f(P)f(S)f(Q)f(R)$)
$= 50 \times 4 = 200$ square units.

Blunders (-3)

B1 Error in establishing area of square

B2 Error in det f^{-1}

B3 Incomplete answer

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Area of square $4r^2$ and stop

MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAEILGE

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronnadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d'iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú **síos**.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéir ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. 198 marc $\times$ 5% = 9·9 $\Rightarrow$ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, ríomhtar an bónas de réir na foirmle [300 – bunmharc] $\times$ 15%, agus an marc bónais sin a shlánú **síos**. In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

Bunmharc	Marc Bónais
226	11
227 – 233	10
234 – 240	9
241 – 246	8
247 – 253	7
254 – 260	6
261 – 266	5
267 – 273	4
274 – 280	3
281 – 286	2
287 – 293	1
294 – 300	0