AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA

LEAVING CERTIFICATE EXAMINATION, 1998

MATHEMATICS – HIGHER LEVEL – PAPER I (300 marks)

THURSDAY, 11 JUNE – MORNING 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each).

Marks may be lost if necessary work is not clearly shown or you do not indicate where a calculator has been used.

1. **(a)** Solve for x and y:

$$\frac{2x-5}{3} + \frac{y}{5} = 6$$

$$\frac{3x}{10} + 2 = \frac{3y-5}{2}.$$

(b) If $(2x - 1)$ is a factor of the polynomial

$$P(x) = 2x^3 - 5x^2 - kx + 3,$$

find the value of k.

Find the other two factors of $P(x)$.

(c) If the quadratic equation $ax^2 + bx + c = 0$ has equal roots, solve for x in terms of a and b, where $a, b, c \in \mathbf{R}$.

By letting $x = 3^y$, write

$$t3^y + 3^{-y} = 3$$

as a quadratic equation in x, where $t \in \mathbf{R}$ and $t \neq 0$.
Find the value of t for which this equation has equal roots.

Assuming this value of t, solve the equation

$$t3^y + 3^{-y} = 3.$$

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2. (a) Solve for x: $|x-4|<5$.

(b) If α and β are the roots of the equation

$$x^2 - 6x + 2 = 0, \qquad \alpha > 0, \beta > 0,$$

find $\alpha\beta$ and $\alpha + \beta$.

Factorise $\alpha^3 + \beta^3$.

Find the value of $\alpha^3 + \beta^3$.

(c) Show for all real numbers $a, b > 0$, that

$$a + b \geq 2\sqrt{ab}.$$

Show that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{(a+b)^2}{ab}$.

Deduce that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

3. (a) Express $\sqrt{3}+i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.

(b) If $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$, find AB.

Show that $B^{-1}AB$ is of the form $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, where $p, q \in \mathbf{N}_0$.

(c) Let $z = \cos\theta + i\sin\theta$.

Express $\frac{2}{1+z}$ in the form $1 - i\tan(k\theta)$, $k \in \mathbf{Q}$ and $z \neq -1$.

4. (a) Find the sum to infinity of the geometric series

$$1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots$$

(b) If for all integers n,

$$u_n = 3 + n(n-1)^2,$$

show that

$$u_{n+1} - u_n = 3n^2 - n.$$

(c) Show that $\frac{1}{4n^2 - 1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$.

Let $u_n = \frac{1}{4n^2 - 1}$.

Find $\sum_{n=1}^{\infty} u_n$.

Find the least value of r such that

$$\sum_{n=1}^{r} u_n > \frac{99}{100} \sum_{n=1}^{\infty} u_n, \quad r \in \mathbf{N}.$$

5. **(a)** Find the value of the term which is independent of x in the expansion of

$$\left(x^2 - \frac{1}{x}\right)^9.$$

(b) Solve

$$\log_5(x - 2) = 1 - \log_5(x - 6), \qquad x \in \mathbf{R}, x > 6.$$

(c) Let $u_n = (1 + x)^n - 1 - nx$ for $n \in \mathbf{N}_0$, $x \in \mathbf{R}$ and $x > -1$ and where $u_n = u_n(x)$.

Show that

$$u_{n+1} \geq u_n$$

(i) when $x = 0$

(ii) when $x > 0$

(iii) when $-1 < x < 0$.

Show that $u_2 \geq 0$.

Hence, or otherwise, deduce that

$$(1 + x)^n \geq 1 + nx, \quad x > -1.$$

6. **(a)** Differentiate **(i)** $(1 + 3x)^2$ **(ii)** $3e^{4x+1}$.

(b) Find the value of the constant k if $y = kx^2$ is a solution of the equation

$$x\frac{dy}{dx} + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + y = 0,$$

where $x \in \mathbf{R}$ and $k \neq 0$.

(c) Given that $f(x) = \dfrac{x}{x+2}$, $x \in \mathbf{R}$ and $x \neq -2$,

find the equations of the asymptotes of the graph of $f(x)$.

Prove that the graph of $f(x)$ has no turning points or points of inflection.

Find the range of values of x for which $f'(x) \leq 1$, where $f'(x)$ is the derivative of $f(x)$.

7. (a) Let $\theta = 5t^3 - 2t^2$,

where t is in seconds and θ is in radians.

Find the rate of change of θ when $t = 2$ seconds.

(b) The parametric equations of a curve are

$$x = \frac{1 + \sin t}{\cos t}, \qquad y = \frac{1 + \cos t}{\sin t}, \qquad 0 < t < \frac{\pi}{2}.$$

Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Find the slope of the tangent to the curve at the point where $t = \tan^{-1}\left(\frac{3}{4}\right)$.

(c) Let $f(x) = x^3 - kx^2 + 8$, $k \in \mathbf{R}$ and $k > 0$.

Show that the coordinates of the local minimum point of $f(x)$ are $\left(\frac{2k}{3}, 8 - \frac{4k^3}{27}\right)$.

Taking $x_1 = 3$ as the first approximation of one of the roots of $f(x) = 0$, the Newton-Raphson method gives the second approximation as $x_2 = \frac{10}{3}$.
Find the value of k.

8. (a) Find (i) $\int (x^2 + 3)dx$ (ii) $\int \frac{1}{x^2} dx$.

(b) Evaluate (i) $\int_2^3 \frac{x-2}{x^2 - 4x + 5} dx$ (ii) $\int_0^{\pi/4} (\cos x + \sin x)^2 dx$.

(c) Find the area of the bounded region enclosed by the line $y = 2x - 1$, the line $x = 4$ and the curve $y = \frac{1}{x}$, where $x > 0$.