Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate 2012

Marking Scheme

Mathematics
(Project Maths – Phase 1)

Higher Level

Contents

Introduction

The Higher Level Mathematics examination for candidates in the initial schools for *Project Maths* shared a common Paper 2 with the examination for all other candidates. The marking scheme used for the shared content was identical for the two groups.

This document contains the complete marking scheme for both papers for the candidates in all schools other than the initial schools. That is, it contains the marking scheme for Phase 1 of Project Maths.

Readers should note that, as with all marking schemes used in the state examinations, the detail required in any answer is determined by the context and the manner in which the question is asked, and by the number of marks assigned to the question or part. Requirements and mark allocations may vary from year to year.

Marking scheme for Paper 1

GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2, …etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5·50 may be written as €5,50.

QUESTION 1

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

1. **(a)** The following equation is true for all x.

$$ax^2 + bx(x-4) + c(x-4) = x^2 + 13x - 20.$$

Find the value of the constants a, b and c.

Equating coefficients	**5 marks**	**Att 2**
Values	**5 marks**	**Att 2**

1 (a)

$$ax^2 + bx^2 - 4bx + cx - 4c = x^2 + 13x - 20.$$

$$x^2(a+b-1) + x(-4b+c-13) + (-4c+20) = 0.$$

$$\therefore \quad -4c+20=0 \quad \Rightarrow \quad c=5.$$

$$\therefore -4b-8=0 \quad \Rightarrow \quad b=-2.$$

$$a-2-1=0 \Rightarrow a=3.$$

$$\therefore \quad a=3,\ b=-2,\ c=5.$$

<u>OR</u>

$$ax^2 + bx(x-4) + c(x-4) = x^2 + 13x - 20$$

True for all values of x

Let $x = 0$: $0+0+c(-4) = 0+0-20$

$$-4c = -20$$

$$c = 5..........(i)$$

Let $x = 1$: $a + b(-3) + c(-3) = 1+13-20$

$$a - 3b - 3c = -6$$

$$a - 3b - 15 = -6$$

$$a - 3b = 9........(ii)$$

Let $x = 2$: $a(2)^2 + 2b(-2) + c(-2) = 4+26-20$

$$4a - 4b - 2c = 10$$

$$4a - 4b - 10 = 10$$

$$4a - 4b = 20$$

$$a - b = 5............(iii)$$

$$\begin{array}{lrl} (ii): & a-3b &= 9 \\ (iii): & \underline{a-b} &\underline{= 5} \\ & -2b &= 4 \\ & b &= -2 \end{array}$$

$$\begin{array}{lrl} (iii): & a-b &= 5 \\ & a+2 &= 5 \\ & a &= 3 \end{array}$$

Blunders (-3)
B1 Not like-to-like when equating coefficients
B2 Indices

Slips (-1)
S1 Numerical

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

1. (b)	The function $f(x) = x^3 - 2x^2 - 5x + 6$ has three integer roots.
(i)	Find the three roots.
(ii)	Find a cubic equation whose roots are 1 less than the roots of f.

(b)(i) Root	**5 marks**	**Att 2**
Quadratic factor	**5 marks**	**Att 2**
Roots	**5 marks**	**Att 2**

1 (b) (i)

$f(1) = 1 - 2 - 5 + 6 = 0 \quad \Rightarrow x = 1 \quad \Rightarrow \quad (x-1)$ is a factor.

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x-1 & x^3 - 2x^2 - 5x + 6 \\ & \underline{x^3 - x^2} \\ & \quad -x^2 - 5x + 6 \\ & \quad \underline{-x^2 + x} \\ & \qquad -6x + 6 \\ & \qquad \underline{-6x + 6} \\ & \qquad\quad 0 \end{array}$$

$\therefore \quad x^2 - x - 6 = 0 \quad \Rightarrow \quad (x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3, x = -2.$

The three roots are 1, 3, and – 2.

OR

(b)(i) Root	**5 marks**	**Att 2**
Quadratic factor	**5 marks**	**Att 2**
Roots	**5 marks**	**Att 2**

1 (b) (i)

$f(x) = x^3 - 2x^2 - 5x + 6$

$f(1) = 1 - 2 - 5 + 6 = 0$

$\Rightarrow (x-1)$ is a factor.

$f(x) = x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 + ax - 6)$

$x^3 - (-a+1)x^2 - (a+6)x + 6 = x^3 - 2x^2 - 5x + 6$

Equating coefficients:

$-a + 1 = 2$

$a = -1$

$f(x) = (x-1)(x^2 - x - 6)$

$= (x-1)[(x+2)(x-3)]$

$\Rightarrow x = 1$ or $x = -2$ or $x = 3$

The three roots are 1, 3, and – 2.

OR

(b)(i) 1st root	**5 marks**	**Att 2**
2nd root	**5 marks**	**Att 2**
3rd root	**5 marks**	**Att 2**

1 (b) (i)

$f(x) = x^3 - 2x^2 - 5x + 6$

$6 \Rightarrow (\pm 1), (\pm 2), (\pm 3), (\pm 6)$

$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0$

$x = 1$ is a root.

$f(2) = (2)^3 - 2(2)^2 - 5(2) + 6$
$= 8 - 8 - 10 + 6$
$\neq 0$

$f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6$
$= -8 - 8 + 10 + 6$
$= 0$

$x = -2$ is a root.

$f(3) = (3)^3 - 2(3)^2 - 5(3) + 6$
$= 27 - 18 - 15 + 6$
$= 0$

$x = 3$ is a root.

Roots: {1, -2, 3}

Blunders (-3)

B1 Test for root
B2 Deduction of factor from root, or no deduction
B3 Indices
B4 Factors (once only)
B5 Root formula (once only)
B6 Deduction of root from factor, or no deduction
B7 Not like-to-like when equating coefficients

Slips (-1)
S1 Numerical
S2 Not changing sign when subtracting in division

Attempts
A1 Att2 once only for testing of incorrect values if no other work of merit

Worthless
W1 $x(x^2 - 2x - 5) = -6$, with or without further work

NOTE: if there is a remainder after division, or incomplete division, candidates can only get Att at most for remaining factor and roots.

(b) (ii) **5 marks** **Att 2**

1 (b) (ii)

New cubic equation has roots 0, 2 and −3.

$\therefore\ x(x-2)(x+3)=0 \ \Rightarrow\ x(x^2+x-6)=0 \ \Rightarrow\ x^3+x^2-6x=0.$

Blunders (-3)

B1 New roots
B2 Indices
B3 Factor from roots, or no factor

Slips (-1)

S1 "$\neq$" 0 i.e. not as equation

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

1. (c) (i) Show that $kx-t$ is a factor of $k^3x^3-k^2tx^2+ktx-t^2$, where k and t are non-zero real constants.

(ii) Given any value of $k \neq 0$, find the set of values of t for which the equation $k^3x^3-k^2tx^2+ktx-t^2=0$ has three distinct real roots.

(c)(i) Division **5 marks** **Att 2**
Remainder = 0 **5 marks** **Att 2**

1 (c) (i)

$$\begin{array}{r}
k^2x^2+t \\
(kx-t)\overline{\big)\,k^3x^3-k^2tx^2+ktx-t^2} \\
\underline{k^3x^3-k^2tx^2} \qquad\qquad \\
ktx-t^2 \\
\underline{ktx-t^2} \\
0
\end{array}$$

No remainder;

$\therefore\ (kx-t)$ is a factor.

OR

(c) (i) $\frac{t}{k}$ **5 marks** **Att 2**

$f\left(\frac{t}{k}\right)=0$ **5 marks** **Att 2**

1 (c) (i)

If $(kx-t)$ is a factor, $f\left(\frac{t}{k}\right)=0$

$$f(x)=k^3x^3-k^2tx^2+ktx-t^2$$

$$f\left(\frac{t}{k}\right)=k^3\left(\frac{t}{k}\right)^3-k^2t\left(\frac{t}{k}\right)^2+kt\left(\frac{t}{k}\right)-t^2$$

$$=t^3-t^3+t^2-t^2$$

$$=0$$

$\therefore$ $(kx-t)$ is a factor.

OR

(c) (i) Other factor **5 marks** **Att 2**
Finish **5 marks** **Att 2**

1 (c) (i)

If $(kx-t)$ is a factor, other factor is (k^2x^2+bx+t)

$$k^3x^3-k^2tx^2+ktx-t^2=(kx-t)(k^2x^2+bx+t)$$

$$k^3x^3-k^2tx^2+ktx-t^2=k^3x^3+bkx^2+ktx-k^2tx^2-btx-t^2$$

$$=k^3x^3-(k^2t-bk)x^2+(kt-bt)x-t^2$$

Equating coefficients

(1): $k^2t=k^2t-bk$

$bk=0$

$k\neq 0\Rightarrow b=0$

$\Rightarrow f(x)=(kx-t)(k^2x^2+t)$

Blunders (-3)
B1 Indices
B2 Not like-to-like when equating coefficients
B3 Deduction root from factor
B4 Incorrect deduction from *bk*=0, or no deduction

Slips (-1)
S1 Not changing sign when subtracting in division

NOTE: If there is a remainder after division, or incomplete division, candidates can only get Att at most in 2nd part.

(c) (ii) Roots | **5 marks** | **Att 2**
Values of t | **5 marks** | **Att 2**

1 (c) (ii)

$$k^3x^3 - k^2tx^2 + ktx - t^2 = (kx - t)(k^2x^2 + t) = 0.$$

$$\therefore \quad x = \frac{t}{k} \quad \text{or} \quad x = \pm\sqrt{\frac{-t}{k^2}} = \pm\frac{\sqrt{-t}}{|k|}$$

For real roots, $t \le 0$.

For three distinct real roots, $t < 0$ and $t \ne -1$.

Blunders (-3)

B1 Deduction root from factor

B2 Not 3 roots

B3 Omission of $t \ne -1$

NOTE: Accept $\pm\frac{\sqrt{-t}}{k}$

QUESTION 2

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**25 (5, 5, 5, 5, 5) marks**	**Att (2, 2, 2, 2, 2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

2. (a)

Solve for x: $\sqrt{2x+3} = 2x-3$, where $x \in \mathbf{R}$.

Quadratic	**5 marks**	**Att 2**
Solution	**5 marks**	**Att 2**

2 (a)

$$\sqrt{2x+3} = 2x-3$$

$$2x+3 = 4x^2-12x+9$$

$$4x^2-14x+6=0$$

$$2x^2-7x+3=0$$

$$(2x-1)(x-3)=0 \quad \Rightarrow \quad x=3, \quad x=\tfrac{1}{2}.$$

Check $x = 3$: $\sqrt{9} = 3$ ✓

Check $x=\frac{1}{2}$: $\sqrt{4} = -2$ ✗

$\therefore$ Solution is $x = 3$.

Blunders (-3)

B1 Indices

B2 Expansion of $(2x-3)^2$ once only

B3 Factors once only

B4 Roots formula once only

B5 Deduction root from factor

B6 Excess values

Slips (-1)

S1 Numerical

Attempts

A1 $x = 3$ and no other work merits Att2 only

A2 $x = 3$ by trial and error merits Att2 only

Part (b) **25 (5, 5, 5, 5, 5) marks** **Att (2, 2, 2, 2, 2)**

2. **(b)** α and β are the roots of the equation $x^2 - 2x + 5 = 0$.

(i) Find the value of $\alpha^2 + \beta^2$.

(ii) Find a quadratic equation whose roots are $\alpha + \dfrac{1}{\alpha}$ and $\beta + \dfrac{1}{\beta}$..

(b)(i) Value of sum	**5 marks**	**Att 2**
Value of product	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

2 (b) (i)

$$\alpha + \beta = -\frac{b}{a} = 2 \text{ and } \alpha\beta = \frac{c}{a} = 5.$$

$$(\alpha + \beta)^2 = 4 \Rightarrow \alpha^2 + \beta^2 = 4 - 2\alpha\beta = 4 - 10 = -6.$$

(b)(ii) Values of sum and product	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

2 (b) (ii)

$$\text{Sum of roots } = \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta}$$

$$= \alpha + \beta + \frac{\alpha + \beta}{\alpha\beta}$$

$$= 2 + \frac{2}{5} = \frac{12}{5}.$$

$$\text{Product of roots } = \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$$

$$= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$$

$$= 5 + \frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{1}{5}$$

$$= 5 + \frac{-6}{5} + \frac{1}{5} = 4$$

$$\therefore \text{ equation is } x^2 - \frac{12}{5}x + 4 = 0 \text{ or } 5x^2 - 12x + 20 = 0.$$

<u>OR</u>

(b)(ii) Values of sum and product — 5 marks — Att 2
Finish — 5 marks — Att 2

2 (b) (ii)

New roots: $\alpha + \frac{1}{\alpha} = \frac{\alpha^2+1}{\alpha}$

$\beta + \frac{1}{\beta} = \frac{\beta^2+1}{\beta}$

Sum: $\frac{\alpha^2+1}{\alpha} + \frac{\beta^2+1}{\beta} = \frac{\alpha^2\beta + \beta + \alpha\beta^2 + \alpha}{\alpha\beta}$

$= \frac{(\alpha^2\beta + \alpha\beta^2) + (\alpha+\beta)}{\alpha\beta}$

$= \frac{\alpha\beta(\alpha+\beta) + (\alpha+\beta)}{\alpha\beta}$

$= \frac{(5\times 2)+2}{5}$

$= \frac{12}{5}$

Product: $\left(\frac{\alpha^2+1}{\alpha}\right)\left(\frac{\beta^2+1}{\beta}\right) = \frac{\alpha^2\beta^2 + \beta^2 + \alpha^2 + 1}{\alpha\beta}$

$= \frac{(\alpha\beta)^2 + (\alpha^2+\beta^2) + 1}{\alpha\beta}$

$= \frac{(5)^2 + (-6) + 1}{5}$

$= 4$

$\therefore$ equation is $x^2 - \frac{12}{5}x + 4 = 0$ or $5x^2 - 12x + 20 = 0$.

Blunders (-3)

B1 Indices
B2 Incorrect statement
B3 Incorrect sum
B4 Incorrect product
B5 Factors once only

Slips (-1)

S1 Numerical
S2 "$\neq$" 0 i.e. not as equation

Part (c)	15 (5, 5, 5) marks	Att (2, 2, 2)

2. **(c)** **(i)** Show that if x is a positive real number, then $x + \dfrac{1}{x} \geq 2.$

(ii) Show that is x is a negative real number, then $x + \dfrac{1}{x} \leq -2.$

(iii) Show that, for all $x \in \mathbf{R} \setminus \{0\}$, $\left| x^3 + \dfrac{1}{x^3} \right| \geq 2.$

(c) (i)	5 marks	Att 2

2 (c) (i)

$$x + \frac{1}{x} \geq 2 \quad \Leftrightarrow \quad x^2 - 2x + 1 \geq 0 \text{, since } x > 0 .$$

i.e., $(x-1)^2 \geq 0$, which is true.

(c) (ii)	5 marks	Att 2

2 (c) (ii)

$$x + \frac{1}{x} \leq -2 \quad \Leftrightarrow \quad x^2 + 2x + 1 \geq 0 \text{, since } x < 0 .$$

i.e., $(x+1)^2 \geq 0$, which is true.

(c) (iii)	5 marks	Att 2

2 (c) (iii)

If x is positive, then x^3 is positive, so part **(i)** implies $x^3 + \dfrac{1}{x^3} \geq 2$.

If x is negative, then x^3 is negative, so part **(ii)** implies $x^3 + \dfrac{1}{x^3} \leq -2$.

So, the result holds in both cases.

OR

$$\left| x^3 + \frac{1}{x^3} \right| = \left| \left(x + \frac{1}{x} \right) \left(x^2 - 1 + \frac{1}{x^2} \right) \right| .$$

From parts **(i)** and **(ii)**, $\left| x + \dfrac{1}{x} \right| \geq 2 \quad \Rightarrow \quad x^2 + \dfrac{1}{x^2} + 2 \geq 4.$

$$\therefore \; x^2 + \frac{1}{x^2} \geq 2 \quad \Rightarrow \quad x^2 + \frac{1}{x^2} - 1 \geq 1.$$

$$\therefore \; \left| x^3 + \frac{1}{x^3} \right| \geq 2.$$

OR

$$\left| x^3 + \frac{1}{x^3} \right| \geq 2$$

$$\left(x^3 + \frac{1}{x^3} \right)^2 \geq 4$$

$$x^6 + 2 + \frac{1}{x^6} \geq 4$$

$$x^6 - 2 + \frac{1}{x^6} \geq 0$$

Multiply across by x^6

$$x^{12} - 2x^6 + 1 \geq 0$$

$$\left(x^6 - 1\right)^2 \geq 0$$

True

Blunders (-3)

B1 $(x + \frac{1}{x})^p$ incorrect once only

B2 Inequality sign

B3 Incorrect deduction or no deduction

B4 Factors

B5 Modulus

Slips

S1 Not ‘≥’

QUESTION 3

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

3. **(a)** Verify that $z = 2 - 3i$ satisfies the equation $z^3 - z^2(2-3i) + z - 2 + 3i = 0$, where $i^2 = -1$.

(a) **10 marks** **Att 3**

3 (a)

$z = 2 - 3i \quad \Rightarrow \quad (2-3i)^3 - (2-3i)^2(2-3i) + 2 - 3i - 2 + 3i = 0$

$(2 - 3i)^3 - (2 - 3i)^3 + 2 - 3i - 2 + 3i = 0$

$\therefore \quad z = 2 - 3i$ is a solution.

<u>OR</u>

$z = 2 - 3i$

$z^3 - z^2(2-3i) + z - 2 + 3i = 0$

$z^3 - z^2(z) + z - (2-3i) = 0$

$z^3 - z^3 + z - z = 0$

True $\therefore \quad z = 2 - 3i$ is a solution.

<u>OR</u>

$z = 2 - 3i$

$z^2 = (2-3i)^2 = 4 - 12i + 9i^2 = -5 - 12i$

$z^3 - z^2(2-3i) + z - 2 + 3i$

$= (-5-12i)(2-3i) - (-5-12i)(2-3i) + (2-3i) - (2-3i)$

$= 0 - 0$

$= 0$

True $\therefore \quad z = 2 - 3i$ is a solution.

Blunders (-3)

B1 i^2
B2 z^2 once only
B3 z^3 once only
B4 Incorrect deduction or no deduction

Part (b) **20 (10, 5, 5) marks** **Att (3, 2, 2)**

3. (b) Let $A = \begin{pmatrix} 2y & y \\ x^2 & x \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$, where $x, y \in \mathbf{R}$.

(i) Find AB in terms of x and y.

(ii) Solve for x and y the equation $AB = \begin{pmatrix} -4 & 5 \\ 15 & -24 \end{pmatrix}$.

(b) (i) **10 marks** **Att 3**

3 (b) (i)

$$AB = \begin{pmatrix} 2y & y \\ x^2 & x \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4y & -5y \\ x^2+2x & -3x^2+x \end{pmatrix}.$$

Blunders (3)

B1 Each incorrect element

(b)(ii) Equations **5 marks** **Att 2**

Solutions **5 marks** **Att 2**

3 (b) (ii)

$$AB = \begin{pmatrix} 4y & -5y \\ x^2+2x & -3x^2+x \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 15 & -24 \end{pmatrix}$$

$4y = -4$ and $-5y = 5$, so $y = -1$.

$x^2 + 2x = 15$ and $-3x^2 + x = -24$.

$\therefore\ x^2 + 2x - 15 = 0$ and $3x^2 - x - 24 = 0$.

$(x-3)(x+5) = 0$ and $(3x+8)(x-3) = 0$.

$\therefore\ x = 3$.

Blunders (-3)

B1 Not like-to-like

B2 Extra values of x

B3 Factors

B4 Deduction root from factor or no deduction

B5 Indices

<u>Must state x=3</u>

A1 No common value

NOTE: 2 quadratic equations needed in b(ii), otherwise Att2 + Att2

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

3. **(c)** z is a complex number such that $z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

(i) Find the two possible values of z.

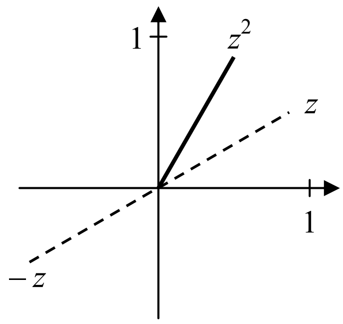
(ii) On the Argand diagram the points representing $-z$, z and $z^2 + k$ are collinear, where $k \in \mathbf{R}$. Find the value of k.

(c)(i) z^2 in polar form **5 marks** **Att 2**

Two values of z **5 marks** **Att 2**

3 (c) (i)

$$r = \left|z^2\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1. \quad \tan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}. \quad \therefore\ \theta = \frac{\pi}{3}.$$



$$z^2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$z = \left[\cos\left(\frac{\pi}{3}+2n\pi\right) + i\sin\left(\frac{\pi}{3}+2n\pi\right)\right]^{\frac{1}{2}}$$

$$= \cos\left(\frac{\pi}{6}+n\pi\right) + i\sin\left(\frac{\pi}{6}+n\pi\right)$$

$$n = 0 \quad \Rightarrow \quad z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \quad = \quad \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

$$n = 1 \quad \Rightarrow \quad z = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} \quad = \quad -\frac{\sqrt{3}}{2} - \frac{1}{2}i.$$

$$\therefore\ z = \pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right).$$

Blunders (-3)

B1 Formula for De Moivre once only
B2 Application of De Moivre
B3 Argument
B4 Modulus
B5 Polar formula once only
B6 i
B7 Not two values of z
B8 Left in polar form

<u>OR</u>

(c) (i) Equations	**5 marks**	**Att 2**
Two values of z	**5 marks**	**Att 2**

3 (c) (i)

$$z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = (a+bi)^2$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i = (a^2 - b^2) + (2ab)i$$

Equating coefficients: (i): $a^2 - b^2 = \frac{1}{2}$

(ii): $2ab = \frac{\sqrt{3}}{2} \Rightarrow b = \frac{\sqrt{3}}{4a}$

(i): $a^2 - b^2 = \frac{1}{2}$

$$a^2 - \left(\frac{\sqrt{3}}{4a}\right)^2 = \frac{1}{2}$$

$$a^2 - \frac{3}{16a^2} = \frac{1}{2}$$

Let $p = a^2 \Rightarrow p \in \mathbf{R}$

$$p - \frac{3}{16p} = \frac{1}{2}$$

$$16p^2 - 3 = 8p$$

$$16p^2 - 8p - 3 = 0$$

$$(4p+1)(4p-3) = 0$$

$$\Rightarrow p = -\frac{1}{4} \text{ or } p = \frac{3}{4}$$

But, $p = a^2 \neq -\frac{1}{4}$

$$\Rightarrow a^2 = \frac{3}{4}$$

$$a = \pm\frac{\sqrt{3}}{2}$$

(ii): $b = \frac{\sqrt{3}}{4a}$

$$a = \frac{\sqrt{3}}{2} \Rightarrow b = \frac{1}{2}; \quad a = -\frac{\sqrt{3}}{2} \Rightarrow b = -\frac{1}{2}$$

$$a + bi = \pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

Blunders (-3)

B1 Expansion of $(a+ib)^2$
B2 Indices
B3 i
B4 Not like-to-like
B5 Factors once only
B6 Quadratic formula once only
B7 Excess values (not real)
B8 Not two values of z
B9 Incorrect deduction root from factor or no deduction

(c) (ii)Arguments equated	**5 marks**	**Att2**
k	**5 marks**	**Att2**

3 (c) (ii)

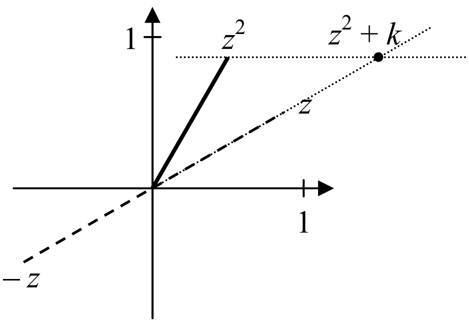
$$z^2+k=\left(\tfrac{1}{2}+k\right)+\tfrac{\sqrt{3}}{2}i$$

Collinear $\Rightarrow$ $\arg(z^2+k)=\arg(z)$

$$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}+k}=\frac{1}{\sqrt{3}}$$

$$\tfrac{3}{2}=\tfrac{1}{2}+k$$

$$k=1$$



OR

(c)(ii) *f*(*z*) = *lz*	**5 marks**	**Att2**
k	**5 marks**	**Att2**

3(c)(ii)

z^2+k is a multiple of z, so $z^2+k=lz$.

Equate imaginary parts to give $\dfrac{\sqrt{3}}{2}=\dfrac{l}{2} \quad\Rightarrow\quad l=\sqrt{3}$.

Then equate real parts to give $\tfrac{1}{2}+k=\tfrac{3}{2} \;\Rightarrow\; k=1$.

OR

(c) (ii) Triangle area **5 marks** **Att2**
k **5 marks** **Att2**

3(c)(ii)

Since collinear, the triangle with vertices $z,\ -z$ and z^2+k has area $=0$.

Vertices are $\left(\dfrac{-\sqrt{3}}{2},\dfrac{-1}{2}\right),\ \left(\dfrac{\sqrt{3}}{2},\dfrac{1}{2}\right),\ \left(\dfrac{1}{2}+k,\dfrac{\sqrt{3}}{2}\right).$

Translating vertex $\left(\dfrac{-\sqrt{3}}{2},\dfrac{-1}{2}\right)$ to $(0,0)$ gives vertices

$(0,0),\ (\sqrt{3},1),\ \left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}+k,\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\right).$

$$\therefore\ \frac{1}{2}\left|\left(\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)-(1)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}+k\right)\right|=0$$

$$\Rightarrow\ \left|\frac{3}{2}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}-\frac{1}{2}-k\right|=0$$

$$\Rightarrow\ |1-k|=0.$$

$$\therefore\ k=1.$$

<u>OR</u>

(c) (ii) Equation of line **5 marks** **Att2**

k **5 marks** **Att2**

3(c)(ii)

Slope of line through z and $(-z)$ is

$$m = \frac{\frac{1}{2}+\frac{1}{2}}{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Equation of line through $z\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ with slope $m = \frac{1}{\sqrt{3}}$ is

$$\left(y - \frac{1}{2}\right) = \frac{1}{\sqrt{3}}\left(x - \frac{\sqrt{3}}{2}\right)$$

$$y\sqrt{3} - \frac{\sqrt{3}}{2} = x - \frac{\sqrt{3}}{2}$$

$$y\sqrt{3} = x$$

$$y = \frac{x}{\sqrt{3}}$$

$$z^2 + k = \left(\frac{1}{2} + k\right) + \frac{\sqrt{3}}{2}i$$

When $(z^2 + k)$ is on the line $y = \frac{x}{\sqrt{3}}$,

$$\frac{\sqrt{3}}{2} = \frac{\frac{1}{2}+k}{\sqrt{3}}$$

$$3 = 1 + 2k$$

$$2 = 2k$$

$$k = 1$$

Blunders (-3)

B1 Argument

B2 Incorrect $z^2 + k$

B3 $\left(z^2 + k\right)$ not a multiple of z

B4 Not like-to-like

B5 Formula for area of triangle

B6 Area of triangle $\neq 0$

B7 Translating points

B8 Indices

B9 Slope

B10 Equation of line

B11 Points not collinear

QUESTION 4

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

4. **(a)** $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are three consecutive terms of an arithmetic sequence, where $a, b, c \in \mathbf{R}\backslash\{0\}$.
Express b in terms of a and c. Give your answer in its simplest form.

Statement AP **5 marks** **Att 2**
Value of *b* **5 marks** **Att 2**

4 (a)

$$\frac{1}{b} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) = \frac{a+c}{2ac}. \quad \therefore\ b = \frac{2ac}{a+c}.$$

Blunders (-3)
B1 Definition of AP
B2 Algebra
B3 Answer not in simplest form

Worthless
W1 Geometric sequence
W2 Puts in values for *a, b, c*

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

4. **(b)** **(i)** Show that $\dfrac{1}{\sqrt{r+1}+\sqrt{r}} = \sqrt{r+1} - \sqrt{r}$, for $r \geq 0$.

(ii) Find $\displaystyle\sum_{r=1}^{n} \frac{1}{\sqrt{r+1}+\sqrt{r}}$.

(iii) Evaluate $\displaystyle\sum_{r=1}^{99} \frac{1}{\sqrt{r+1}+\sqrt{r}}$.

(b) (i) **5 marks** **Att 2**

4 (b) (i)

$$\frac{1}{\sqrt{r+1}+\sqrt{r}} = \frac{\sqrt{r+1}-\sqrt{r}}{\left(\sqrt{r+1}+\sqrt{r}\right)\left(\sqrt{r+1}-\sqrt{r}\right)}$$

$$= \frac{\sqrt{r+1}-\sqrt{r}}{r+1-r} = \sqrt{r+1} - \sqrt{r}.$$

(b)(ii) Set up cancellation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

4 (b) (ii)

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r+1}+\sqrt{r}} = \sum_{r=1}^{n} \left(\sqrt{r+1}-\sqrt{r}\right).$$

$$\begin{aligned}
u_1 &= \cancel{\sqrt{2}} - 1 \\
u_2 &= \cancel{\sqrt{3}} - \cancel{\sqrt{2}} \\
u_3 &= \cancel{\sqrt{4}} - \cancel{\sqrt{3}} \\
\ldots\ & \quad \ldots\ldots. \\
\ldots\ & \quad \ldots\ldots.. \\
u_{n-2} &= \cancel{\sqrt{n-1}} - \cancel{\sqrt{n-2}} \\
u_{n-1} &= \cancel{\sqrt{n}} - \cancel{\sqrt{n-1}} \\
u_n &= \sqrt{n+1} - \cancel{\sqrt{n}} \\
S_n &= \sqrt{n+1} - 1
\end{aligned}$$

(b) (iii)	**5 marks**	**Att 2**

4 (b) (iii)

$$\sum_{r=1}^{99} \frac{1}{\sqrt{r+1}+\sqrt{r}} = \sqrt{100} - 1 = 9.$$

Blunders (-3)
B1 Not conjugate
B2 Indices
B3 Cancellation must be shown or implied
B4 Term omitted
B5 Gets S_r

Slips (-1)
S1 Numerical

NOTE: Must show two terms at start and one term at finish, or vice versa.

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

4. (c)	a, b and c are consecutive terms in a geometric sequence, where $a+b \neq 0$ and $b+c \neq 0$. Show that $\dfrac{2ab}{a+b}$, b and $\dfrac{2bc}{b+c}$ are consecutive terms in an arithmetic sequence.

$\dfrac{2ab}{a+b}$ **in terms of** r	**5 marks**	**Att 2**
$\dfrac{2bc}{b+c}$ **in terms of** r	**5 marks**	**Att 2**
Definition of arithmetic sequence	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

4 (c)

As a, b, c are in geometric sequence, then let $b = ar$ and $c = ar^2$.

$$\frac{2ab}{a+b} = \frac{2a^2r}{a+ar} = \frac{2ar}{1+r}.$$

$$\frac{2bc}{b+c} = \frac{2a^2r^3}{ar+ar^2} = \frac{2ar^2}{1+r}.$$

Arithmetic sequence if and only if

$$b = \frac{1}{2}\left[\frac{2ab}{a+b} + \frac{2bc}{b+c}\right] = \frac{1}{2}\left[\frac{2ar}{1+r} + \frac{2ar^2}{1+r}\right] = \frac{ar(1+r)}{1+r} = ar = b.$$

True. $\therefore$ An arithmetic sequence.

<u>OR</u>

Value of b^2	**5 marks**	**Att 2**
Definition of arithmetic sequence	**5 marks**	**Att 2**
Cross-multiplication	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

4 (c)

a, b, c in geometric sequence

$$\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$$

To show: $\dfrac{2ab}{a+b}, b, \dfrac{2bc}{b+c}$ in arithmetic sequence

$$\left(b - \frac{2ab}{a+b}\right) = \left(\frac{2bc}{b+c} - b\right)$$

$$\frac{ba + b^2 - 2ab}{a+b} = \frac{2bc - b^2 - bc}{b+c}$$

$$\frac{b^2 - ab}{a+b} = \frac{bc - b^2}{b+c}$$

$$b(b-a)(b+c) = b(c-b)(a+b)$$

$$b^2 - ba + bc - ac = ac - ab + bc - b^2$$

$$2b^2 = 2ac$$

$$b^2 = ac$$

Blunders (-3)

B1 Definition of geometric sequence
B2 Definition of arithmetic sequence
B3 Indices

QUESTION 5

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

5. **(a)** Solve for $x \in \mathbf{R}$: $\log_4(2x+6) - \log_4(x-1) = 1$.

$\log f(x) = 1$ **5 marks** **Att 2**

Value of *x* **5 marks** **Att 2**

5 (a)

$$\log_4(2x+6) - \log_4(x-1) = 1.$$

$$\therefore \log_4 \frac{2x+6}{x-1} = 1 \Rightarrow 2x+6 = 4(x-1).$$

$$2x = 10 \Rightarrow x = 5.$$

Blunders (-3)

B1 Log laws

B2 Indices

Worthless

W1 Drops "log"

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

5. **(b)** Consider the binomial expansion of $\left(3x^2 + \frac{1}{2x}\right)^{10}$ in descending powers of x.

(i) Find an expression for the general term.

(ii) Find the coefficient of x^8.

(iii) Show that there is no term independent of x.

(b) (i) General term **5 marks** **Att 2**

5 (b) (i)

$$\text{General term } = t_{r+1} = {}^{10}C_r\left(3x^2\right)^{10-r}\left(\frac{1}{2x}\right)^r = {}^{10}C_r x^{20-3r}(3)^{10-r}\frac{1}{2^r}.$$

(b)(ii) Value of *r* **5 marks** **Att 2**

Coefficient **5 marks** **Att 2**

5 (b) (ii)

$$20 - 3r = 8 \Rightarrow 3r = 12 \Rightarrow r = 4.$$

$$\text{Coefficient of } x^8 = {}^{10}C_4(3)^6\left(\frac{1}{2^4}\right) = \frac{210 \times 729}{16} = \frac{153090}{16} = \frac{76545}{8}.$$

<u>OR</u>

$$\left(3x^2 + \frac{1}{2x}\right)^{10} = \left(3x^2\right)^{10} + \binom{10}{1}\left(3x^2\right)^9\left(\frac{1}{2x}\right) + \binom{10}{2}\left(3x^2\right)^8\left(\frac{1}{2x}\right)^2 + \binom{10}{3}\left(3x^2\right)^7\left(\frac{1}{2x}\right)^3$$

$$+\binom{10}{4}(3x^2)^6\left(\frac{1}{2x}\right)^4+\text{..........}$$

Term with x^8 is $\binom{10}{4}(3x^2)^6\left(\frac{1}{2x}\right)^4$

$$=\binom{10}{4}(3)^6(x^{12})\left(\frac{1}{16}\right)\left(\frac{1}{x^4}\right)$$

$$=\frac{(210).(729)}{16}.x^8$$

$$=\frac{76545}{8}x^8$$

Coefficient $=\dfrac{76545}{8}$

(b) (iii) **5 marks** **Att 2**

5 (b) (iii)

For independent term, power of x is 0.

But $20-3r\neq 0$ as $r\neq\frac{20}{3}$. $\therefore$ No independent term.

Blunders (-3)

B1 General term

B2 Error binomial expansion once only

B3 Indices

B4 Value $\binom{n}{r}$ or no value $\binom{n}{r}$

B5 Correct term in expansion not identified

NOTE: Accept terms from expansion of $f(x)$ to five terms.
Incomplete Pascals Triangle gets Att2 only and no more.

Part (c)	**20 marks (5, 5, 5, 5)**	**Att (2, 2, 2, 2)**

5. (c) (i) Prove that if $k \geq 4$, then $k^2 > 2k+1$.

(ii) Prove by induction that, for all natural numbers $n \geq 4$, $2^n \geq n^2$.

(c) (i)	**5 marks**	**Att 2**

5 (c) (i)

$k \geq 4$.

$\therefore\ (k-1)^2 \geq 3^2 \ \Rightarrow\ k^2 - 2k + 1 \geq 9 \ \Rightarrow\ k^2 \geq 2k + 8.$

$\therefore\ k^2 > 2k + 1.$

OR

$k > 3 \Rightarrow k^2 > 3k = 2k + k > 2k + 1$.

(c)(ii) $P(4)$	**5 marks**	**Att 2**
$P(k)$	**5 marks**	**Att 2**
$P(k+1)$	**5 marks**	**Att 2**

5 (c) (ii)

Assume true for $n = k$.

$\therefore P(k)$: $2^k \geq k^2$.

Test for $n = k+1$.

$P(k+1)$: $2^{k+1} = 2.2^k$

$\geq 2k^2$, by hypothesis $P(k)$.

$2^{k+1} \geq k^2 + k^2$

$2^{k+1} \geq k^2 + 2k + 1$, by part (i)

$2^{k+1} \geq (k+1)^2$

Test for $n = 4$.

$P(4)$: $2^4 \geq 4^2$.

$\therefore$ True for $n = 4$.

So, $P(k+1)$ is true whenever $P(k)$ true.

Since $P(4)$ true, then, by induction, $P(n)$ is true for all natural numbers $n \geq 4$.

Blunders (-3)

B1 Fails to prove case n=4 (not sufficient to say "true for n=4")

B2 Indices

B3 $n \neq 4$

QUESTION 6

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

6. **(a)** Differentiate with respect to x:

(i) $(4x^2-1)^3$.

(ii) $\sin^{-1}\left(\frac{2x}{3}\right)$.

(a) (i) **5 marks** **Att 2**
(a) (ii) **5 marks** **Att 2**

6 (a)

(i) $\frac{d}{dx}\left[(4x^2-1)^3\right] = 3(4x^2-1)^2.8x = 24x(4x^2-1)^2.$

(ii) $\frac{d}{dx}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right] = \frac{1}{\sqrt{9-4x^2}}\times 2.$

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Incorrect a

Attempts
A1 Error in differentiation formula (chain rule)

Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

6. (b) (i) Differentiate $\sqrt{x}$ with respect to x, from first principles.

(ii) Find the equation of the tangent to the curve $y = \sqrt{x}$ at the point $(9, 3)$.

(b) (i) $f(x+h) - f(x)$	**5 marks**	**Att 2**
Multiplication	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

6 (b) (i)

$f(x) = \sqrt{x}$ and $f(x+h) = \sqrt{x+h}$.

$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{\text{x}}$

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

(b) (ii)	**5 marks**	**Att 2**

6 (b) (ii)

Slope of tangent at $(9, 3) = \frac{1}{6}$.

Equation of tangent: $y - 3 = \frac{1}{6}(x-9) \Rightarrow 6y - 18 = x - 9 \Rightarrow 6y = x + 9$.

Blunders (-3)

B1 $f(x+h)$

B2 Indices

B3 No limits shown or implied or no indication $h \to 0$

B4 $h \to \infty$

B5 Conjugate

B6 No left hand side

B7 Slope not $\frac{dy}{dx}$

B8 Equation of tangent

Worthless

W1 Not first principles

NOTE: Can use the binomial theorem to expand $(x+h)^{\frac{1}{2}}$ etc

Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

6. (c) Let f be the function $f : x \to 8x + \sin 4x + 4\sin 2x$, where $x \in \mathbf{R}$.

(i) Find $f'(x)$.

(ii) Express $f'(x)$ in terms of $\cos 2x$.

(iii) Prove that $f(x)$ is increasing for all values of x.

(c) (i)	**5 marks**	**Att 2**
(c) (ii)	**5 marks**	**Att 2**
(c) (iii)	**10 marks**	**Att 3**

6 (c) (i)

$$f'(x) = 8 + 4\cos 4x + 8\cos 2x$$

(ii)

$$f'(x) = 8 + 4\cos 4x + 8\cos 2x = 8 + 4\left(2\cos^2 2x - 1\right) + 8\cos 2x.$$

$$f'(x) = 8\cos^2 2x + 8\cos 2x + 4.$$

(iii)

$$\begin{aligned} f'(x) &= 8\cos^2 2x + 8\cos 2x + 4 \\ &= 8\left(\cos^2 2x + \cos 2x + \tfrac{1}{2}\right) \\ &= 8\left(\left[\cos 2x + \tfrac{1}{2}\right]^2 + \tfrac{1}{4}\right) \\ &\geq 8(0 + \tfrac{1}{4}) \\ &> 0 \end{aligned}$$

Therefore $f(x)$ is increasing

Blunders (-3)

B1 Differentiation

B2 Indices

B3 Trig formula

NOTE: Must show $f'(x) > 0$

QUESTION 7

Part (a)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (b)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

7. (a) Given that $x = 3t^2 - 6t$ and $y = 2t - t^2$, for $t \in \mathbf{R}$, show that $\frac{dy}{dx}$ is constant.

$\frac{dx}{dt}$	**5 marks**	**Att 2**
$\frac{dy}{dt}$	**5 marks**	**Att 2**
$\frac{dy}{dx}$	**5 marks**	**Att 2**

7 (a)

$$\frac{dx}{dt} = 6t - 6, \quad \frac{dy}{dt} = 2 - 2t.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2 - 2t}{6t - 6} = -\frac{1}{3}.$$

Blunders (-3)

B1 Differentiation

B2 Error in getting $\frac{dy}{dx}$

Part (b)	15 (5, 5, 5) marks	Att (2, 2, 2)

7. (b) A curve is defined by the equation $x^2 - 2xy + 3y^2 + 4y = 22$.

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) The points $(-3, 1)$ and $(1, -3)$ are both on this curve.
Show that the tangents at these two points are parallel to each other.

(b)(i) Differentiation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**
(b)(ii)	**5 marks**	**Att 2**

7 (b) (i)

$$2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} + 4\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx}(2x - 6y - 4) = 2x - 2y \;\Rightarrow\; \frac{dy}{dx} = \frac{x-y}{x-3y-2}.$$

(ii)

Slope of tangent at $(-3, 1) = \dfrac{-3-1}{-3-3-2} = \dfrac{1}{2}$.

Slope of tangent at $(1, -3) = \dfrac{1+3}{1+9-2} = \dfrac{1}{2}$.

Equal slopes, therefore parallel tangents.

Blunders (-3)
B1 Differentiation
B2 Incorrect value of x or no value of x in slope
B3 Incorrect value of y or no value of y in slope

Slips (-1)
S1 Numerical

Attempts
A1 Error in differentiation formula
A2 $\frac{dy}{dx} = 2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} + ...$, and uses all $\frac{dy}{dx}$ terms in first 5 marks can get second 5 marks

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

7. **(c)** Let $f(x) = 32x^3 - 48x^2 + 20x - 1$, where $x \in \mathbf{R}$.

(i) Show that f has a root between 0 and 1.

(ii) Take $x_1 = 0{\cdot}5$ as a first approximation to this root.
Use the Newton-Raphson method to find x_2 and x_3, the second and third approximations.

(iii) What can you conclude about all further approximations?

(c) (i)	**5 marks**	**Att 2**
(c) (ii) x_2	**5 marks**	**Att 2**
x_3	**5 marks**	**Att 2**
(c) (iii)	**5 marks**	**Hit or Miss**

7 (c) (i)

$f(0) = -1 < 0$ and $f(1) = 32 - 48 + 20 - 1 = 3 > 0$.
$\therefore f$ has a root between 0 and 1.

(ii)

$f(x) = 32x^3 - 48x^2 + 20x - 1 \Rightarrow f'(x) = 96x^2 - 96x + 20.$

$f(0{\cdot}5) = 1$ and $f'(0{\cdot}5) = -4$

$\therefore x_2 = 0{\cdot}5 - \dfrac{1}{-4} = 0{\cdot}75$

$f(0{\cdot}75) = 0{\cdot}5$ and $f'(0{\cdot}75) = 2$.

$\therefore x_3 = 0{\cdot}75 - \dfrac{0{\cdot}5}{2} = 0{\cdot}5.$

(iii)
All further approximations will continue in the sequence 0·5, 0·75, 0·5, 0·75, …

Blunders (-3)
B1 Newton-Raphson formula once only
B2 Differentiation
B3 Indices
B4 $x_1 \neq 0.5$ once only
B5 Inequality sign
B6 Incorrect value in table (unless an obvious slip)

Slips (-1)
S1 Numerical

Worthless
W1 Incorrect answer and no work

QUESTION 8

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

8. **(a)** Find $\int \left(1 + \cos 2x + e^{3x}\right) dx$.

(a) **10 marks** **Att 3**

8 (a)

$$\int \left(1 + \cos 2x + e^{3x}\right) dx = x + \frac{1}{2}\sin 2x + \frac{1}{3}e^{3x} + c$$

Blunders (-3)
B1 Integration
B2 No 'c'

Attempts
A1 Only 'c' correct, $\Rightarrow$ Att3

Worthless
W1 Differentiation instead of integration

Part (b) **20 (10, 10) marks** **Att (3, 3)**

8. **(b)** **(i)** Evaluate $\int_1^3 \frac{12}{3x-2}\, dx$.

(ii) Evaluate $\int_0^{\frac{\pi}{8}} \sin^2 2x \, dx$.

(b) (i) **10 marks** **Att 3**
(b) (ii) **10 marks** **Att 3**

8 (b) (i)

Let $u = 3x - 2$. $\therefore$ $du = 3dx$.

$$\int_1^3 \frac{12}{3x-2}\, dx = \int_1^7 \frac{4}{u}\, du = 4\left[\log_e u\right]_1^7 = 4\log_e 7.$$

(ii)

$$\int_0^{\frac{\pi}{8}} \sin^2 2x\, dx = \int_0^{\frac{\pi}{8}} \frac{1}{2}(1-\cos 4x)dx = \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{4}\sin 0\right)\right] = \frac{\pi}{16} - \frac{1}{8}.$$

<u>OR</u>

$$\int_0^{\frac{\pi}{8}} \sin^2 2x\, dx = \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]$$ (see formula Page 26)

$$= \frac{1}{2}\left[\left(\frac{\pi}{8} - \frac{\sin\frac{4\pi}{8}}{4}\right) - (0-0)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{8} - \frac{1}{4}\right)$$

$$= \frac{\pi}{16} - \frac{1}{8}$$

Blunders (-3)
B1 Integration
B2 Differentiation
B3 Limits
B4 Incorrect order in applying limits
B5 Not changing limits
B6 Not calculating substituted limits
B7 Trig formula

Slips (-1)
S1 Numerical
S2 Trig value

NOTE: (-3) max. deduction in limits

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

8.(c) The function *f* is given by $f(x) = x^2 + k$, where *k* is a constant.

(i) The tangent to the curve $y = (x)$ at the point $(a, f(a))$ passes through the origin, where $a > 0$. Express *a* in terms of *k*.

(ii) The tangent at $(-a, f(-a))$ also passes through the origin.
Find, in terms of *k*, the area of the region enclosed by these two tangents and the curve.

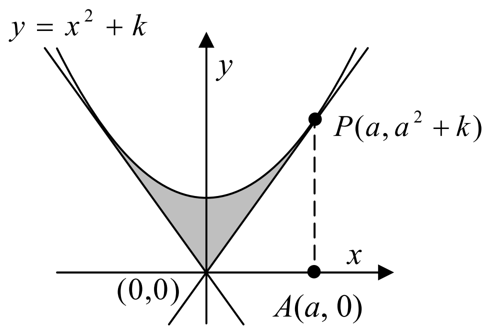
(c)(i) One slope **5 marks** **Att 2**
Express **5 marks** **Att 2**

8 (c) (i)

Point of tangency is $(a, a^2 + k)$ and $f'(x) = 2x \Rightarrow$ slope of tangent $= 2a$.

$$\therefore \frac{a^2 + k}{a} = 2a \ \Rightarrow\ 2a^2 = a^2 + k \ \Rightarrow\ a^2 = k. \ \therefore\ a = \sqrt{k}.$$

<u>OR</u>



Slope of tangent:
Using points: $P(a, a^2 + k)$ and (0,0)

$$m = \frac{a^2 + k}{a} \ \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Using $y = x^2 + k$

$$m = \frac{dy}{dx} = 2x$$

At x=a: m=$2a$…………(ii)

From (i) and (ii):

$$m = 2a = \frac{a^2 + k}{a}$$

$$2a^2 = a^2 + k$$

$$a^2 = k$$

$$a = \sqrt{k}$$

(since a>0)

Blunders (-3)
B1 Blunder point *P*
B2 Blunder slope at *P*
B3 Blunder differentiation

(c)(ii) Area under curve	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

8 (c) (ii)

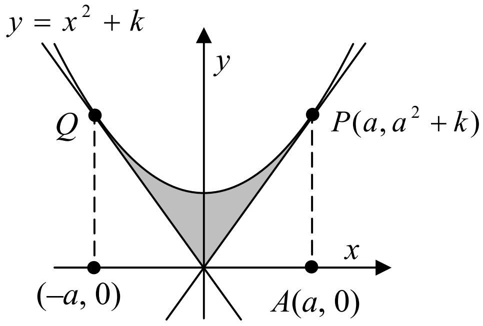
Equation of tangent: $y = mx \;\Rightarrow\; y = 2ax.$

Area $= 2\left[\int_0^a \left(x^2 + a^2\right)dx - \int_0^a 2ax\,dx\right] = 2\left[\frac{1}{3}x^3 + a^2x\right]_0^a - 2\left[ax^2\right]_0^a$

$$= 2\left(\frac{1}{3}a^3 + a^3 - a^3\right) = \frac{2}{3}a^3 = \frac{2k\sqrt{k}}{3}$$

OR

A_1, area between curve and x-axis



$$A_1 = \int_{-a}^{a} y.dx$$

$$= 2\int_0^a (x^2 + k)dx$$

$$= 2\left[\frac{x^3}{3} + kx\right]_0^a$$

$$= 2\left[\left(\frac{a^3}{3} + ka\right) - (0 + 0)\right]$$

$$A_1 = 2\left(\frac{a^3}{3} + ka\right)$$

$A_2 = 2.\text{Area } \Delta OAP$

$$= 2\left[\frac{1}{2}(a)(a^2 + k)\right]$$

$$= a^3 + ak$$

Required area = $A_1 - A_2$

$$= 2\left(\frac{a^3}{3} + ka\right) - \left(a^3 + ak\right)$$

$$= ak - \frac{a^3}{3}$$

$$= k^{\frac{3}{2}} - \frac{k^{\frac{3}{2}}}{3}$$

$$= \frac{2}{3}k^{\frac{3}{2}}$$

Blunders (-3)

B1 Integration
B2 Indices
B3 Calculation point of tangency
B4 Error in formula for area of triangle
B5 Error in area formula
B6 Incorrect order in applying limits
B7 Error with line
B8 Error with curve
B9 Uses $\pi\int y.dx$ for area formula

Attempts

A1 Uses volume formula
A2 Uses y^2 in formula

Worthless

W1 Wrong area formula and no work

2012. M130

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination, 2012

Mathematics
(Project Maths – Phase 1)

Paper 2

Higher Level

Monday 11 June   Morning 9:30 – 12:00

300 marks

Model Solutions – Paper 2

Note: the model solutions for each question are not intended to be exhaustive – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	2 questions

Answer **all eight** questions, as follows:

In Section A, answer:

Questions 1 to 5 and

either Question 6A **or** Question 6B.

In Section B, answer Question 7 and Question 8.

Write your answers in the spaces provided in this booklet. You will lose marks if you do not do so. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

(a) Given the co-ordinates of the vertices of a quadrilateral *ABCD*, describe **three** different ways to determine, using co-ordinate geometry techniques, whether the quadrilateral is a parallelogram.

1. Check whether both pairs of opposite sides have the same slope (slope formula).
2. Check whether both pairs of opposite sides are equal in length (distance formula).
3. Check whether the midpoints of the diagonals coincide (diagonals bisecting each other).
4. Check whether the translation from *A* to *B* is the same as the translation from *D* to *C* [or equivalent.]
5. Check whether a pair of opposite sides have the same slope and are equal in length (slope and distance formulae).
6. Use slopes and the formula for the angle between two lines to check whether both pairs of opposite angles are equal.
7. Use slopes and the formula for the angle between two lines to check whether $|\angle A| + |\angle B| = 180°$, and $|\angle C| + |\angle B| = 180°$. [or equivalent]

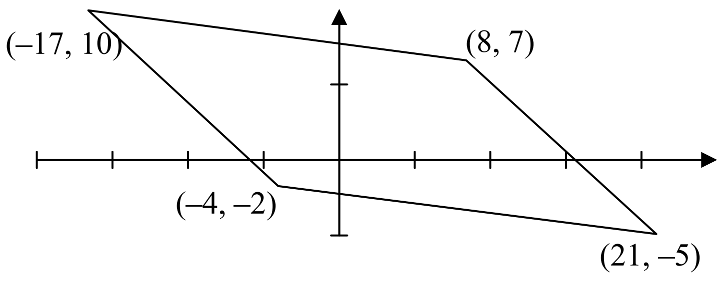
(b) Using **one** of the methods you described, determine whether the quadrilateral with vertices (–4, –2), (21, –5), (8, 7) and (–17, 10) is a parallelogram.

Midpoints of diagonals:

$$\left(\frac{-4+8}{2}, \frac{-2+7}{2}\right) = \left(2, \frac{5}{2}\right)$$

$$\left(\frac{-17+21}{2}, \frac{10-5}{2}\right) = \left(2, \frac{5}{2}\right)$$

Equal $\Rightarrow$ parallelogram.



For other methods: slopes are $\frac{-12}{13}$ and $\frac{-3}{25}$; side-lengths are $\sqrt{313}$ and $\sqrt{634}$, translations are $(x, y) \to (x+25, y-3)$ and $(x, y) \to (x+13, y-12)$, or reverse.

Question 2 **(25 marks)**

The equations of two circles are:

$$c_1 : x^2 + y^2 - 6x - 10y + 29 = 0$$
$$c_2 : x^2 + y^2 - 2x - 2y - 43 = 0$$

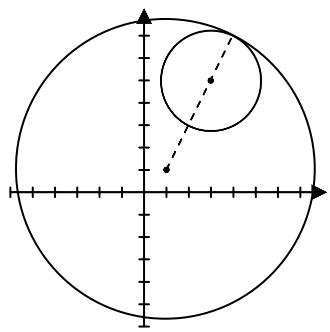
(a) Write down the centre and radius-length of each circle.

$c_1 : (x-3)^2 + (y-5)^2 = 5$ $\therefore$ centre (3, 5); radius $\sqrt{5}$.	$c_2 : (x-1)^2 + (y-1)^2 = 45$ $\therefore$ centre (1, 1); radius $\sqrt{45} = 3\sqrt{5}$.

(b) Prove that the circles are touching.

Distance between centres: $\sqrt{(3-1)^2 + (5-1)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

The distance between the centres is the difference of the radii $\Rightarrow$ circles touch (internally).



(c) Verify that (4, 7) is the point that they have in common.

$$4^2 + 7^2 - 6(4) - 10(7) + 29 = 0 \quad \Rightarrow \quad (4,7) \in c_1$$

$$4^2 + 7^2 - 2(4) - 2(7) - 43 = 0 \quad \Rightarrow \quad (4,7) \in c_2$$

OR

$c_1 - c_2 :\quad x + 2y - 18 = 0 \Rightarrow x = -2y + 18$

$\left(-2y+18\right)^2 + y^2 - 6\left(-2y+18)\right) - 10y + 29 = 0$

$\left(y-7\right)^2 = 0$

$y = 7$

$x = 4$

$\therefore$ (4, 7) common

(d) Find the equation of the common tangent.

Slope from (3, 5) to (4, 7) is: $\dfrac{7-5}{4-3} = 2$

$\therefore$ slope of tangent $= -\frac{1}{2}$.

Equation of tangent:

$$y - 7 = -\tfrac{1}{2}(x-4)$$
$$2y - 14 = -x + 4$$
$$x + 2y - 18 = 0$$

OR

Equation of Tangent: $c_1 - c_2 : x + 2y - 18 = 0$

OR

$$(x-h)(x_1 - h) + (y-k)(y_1 - k) = r^2$$
$$(x-3)(4-3) + (y-5)(7-5) = \left(\sqrt{5}\right)^2$$
$$(x-3) + (y-5)(2) = 5$$
$$x + 2y - 18 = 0$$

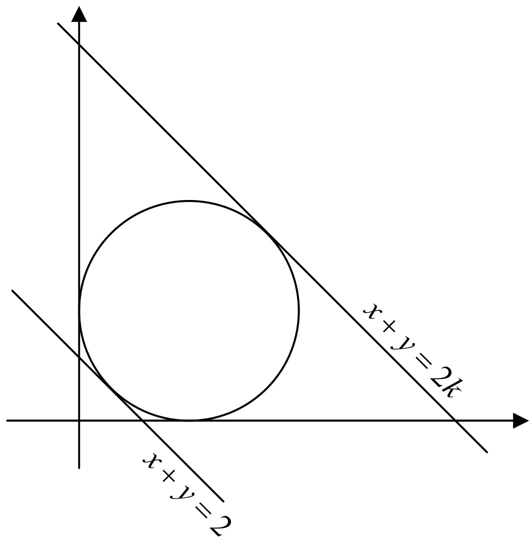
OR

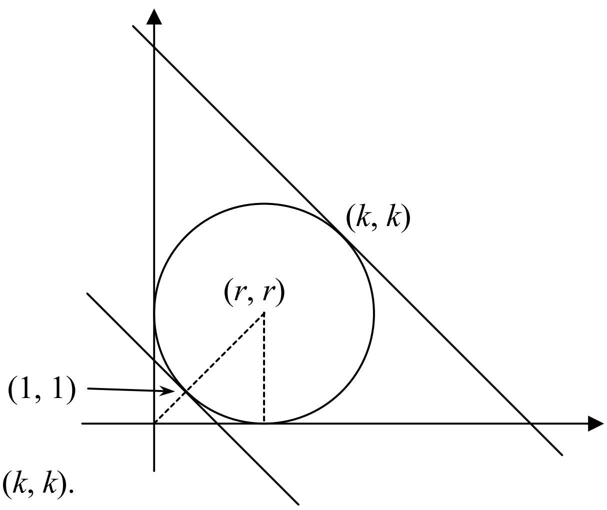
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
$$4x + 7y - 3(x+4) - 5(y+7) + 29 = 0$$
$$x + 2y - 18 = 0$$

Question 3 **(25 marks)**

The circle shown in the diagram has, as tangents, the x-axis, the y-axis, the line $x+y=2$ and the line $x+y=2k$, where $k>1$.

Find the value of k.





$$r^2+r^2=\left(r+\sqrt{2}\right)^2$$
$$2r^2=r^2+2\sqrt{2}r+2$$
$$r^2-2\sqrt{2}r-2=0$$
$$\left(r-\sqrt{2}\right)^2=4$$
$$r=\sqrt{2}+2,\quad (r>0)$$

(r, r) is midpoint of segment from $(1, 1)$ to (k, k).

$$\frac{k+1}{2}=r$$
$$k=2r-1$$
$$k=3+2\sqrt{2}$$

OR

Equation of circle: $(x-r)^2+(y-r)^2=r^2$

The line $x+y=2$ intersects the circle at one point only.

$$y=2-x \Rightarrow (x-r)^2+((2-x)-r)^2=r^2$$
$$\Rightarrow x^2+(2-x)^2+r^2-4r=0$$
$$\Rightarrow 2x^2-4x+(r^2-4r+4)=0$$

One real root $\Rightarrow b^2-4ac=0$

$$\Rightarrow 16-4(2)(r^2-4r+4)=0$$
$$\Rightarrow r=\frac{4\pm\sqrt{16-8}}{2}=2\pm\sqrt{2}$$

But $2-\sqrt{2}$ is too small, so $r=2+\sqrt{2}$

$(1, 1) \rightarrow (2+\sqrt{2},\ 2+\sqrt{2}) \quad \rightarrow (3+2\sqrt{2},\ 3+2\sqrt{2}) = (k, k)$

OR

Centre (r, r)

Perpendicular distance to $x+y-2=0$ equals radius, r.

$$\left|\frac{r+r-2}{\sqrt{2}}\right|=r$$
$$\Rightarrow 2r-2=\pm r\sqrt{2}$$
$$r=\frac{2}{2\pm\sqrt{2}}=2\mp\sqrt{2}$$

But $2-\sqrt{2}$ is too small, so $r=2+\sqrt{2}$

$(1, 1) \rightarrow (2+\sqrt{2},\ 2+\sqrt{2}) \quad \rightarrow (3+2\sqrt{2},\ 3+2\sqrt{2}) = (k, k)$

OR

Having found r as above, find k by setting perpendicular distance from centre (r, r) to $x+y-2k=0$ equal to r:

$$\left|\frac{r+r-2k}{\sqrt{2}}\right|=r$$
$$\Rightarrow 2r-2k=\pm\sqrt{2}r$$
$$\Rightarrow 2\left(2+\sqrt{2}\right)-2k=\pm\sqrt{2}\left(2+\sqrt{2}\right)$$
$$\Rightarrow 4+2\sqrt{2}-2k=\pm\left(2\sqrt{2}+2\right)$$
$$\Rightarrow 2k=4+2\sqrt{2}\pm\left(2\sqrt{2}+2\right)$$
$$\Rightarrow k=2+\sqrt{2}\pm\left(\sqrt{2}+1\right)$$
$$\Rightarrow k=3+2\sqrt{2} \text{ or } 1.$$

$k=1$ corresponds to the lower line, so the answer is $k=3+2\sqrt{2}$

Question 4 **(25 marks)**

A certain basketball player scores 60% of the free-throw shots she attempts. During a particular game, she gets six free throws.

(a) What assumption(s) must be made in order to regard this as a sequence of Bernoulli trials?

Trials are independent of each other.
Probability of success is the same each time.

[Only two outcomes …. (Given)]
[Finite number of throws…… (Given)]

(b) Based on such assumption(s), find, correct to three decimal places, the probability that:

(i) she scores on exactly four of the six shots

$$P(X=4) = {}^6C_4(0{\cdot}6)^4(0{\cdot}4)^2 = 0{\cdot}31104$$
$$= 0{\cdot}311 \text{ to three decimal places.}$$

(ii) she scores for the second time on the fifth shot.

Exactly one success among first four throws, followed by success on fifth:

$$\left({}^4C_1(0{\cdot}6)(0{\cdot}4)^3\right)(0{\cdot}6) = 0{\cdot}09216$$
$$= 0{\cdot}092 \text{ to three decimal places.}$$

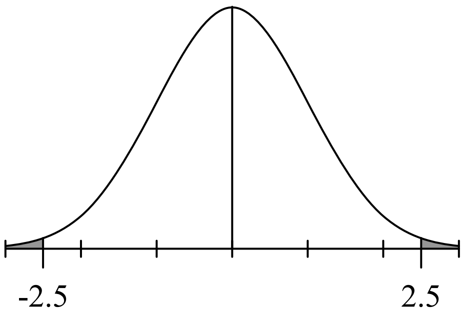
Question 5 **(25 marks)**

A company produces calculator batteries. The diameter of the batteries is supposed to be 20 mm. The tolerance is 0·25 mm. Any batteries outside this tolerance are rejected. You may assume that this is the only reason for rejecting the batteries.

(a) The company has a machine that produces batteries with diameters that are normally distributed with mean 20 mm and standard deviation 0·1 mm. Out of every 10 000 batteries produced by this machine, how many, on average, are rejected?

$$Z = \frac{20{\cdot}25 - 20}{0{\cdot}1} = 2{\cdot}5$$

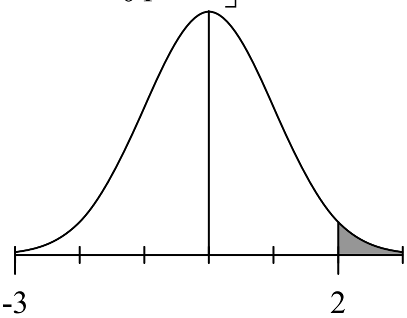
$$\begin{aligned} P\left(\left|X - 20\right| > 0{\cdot}25\right) &= P\left(\left|Z\right| > 2{\cdot}5\right) \\ &= 2\left(1 - P(Z \le 2.5)\right) \\ &= 2\left(1 - 0{\cdot}9938\right) \\ &= 0{\cdot}0124 \end{aligned}$$



Answer = 10 000 × 0·0124 = 124.

(b) A setting on the machine slips, so that the mean diameter of the batteries increases to 20·05 mm, while the standard deviation remains unchanged. Find the percentage increase in the rejection rate for batteries from this machine.

$$\begin{aligned} P(X \le 1{\cdot}975) + P(X \ge 20{\cdot}25) &= P\left[Z \le \frac{19{\cdot}75 - 20{\cdot}25}{0{\cdot}1}\right] + P\left[Z \ge \frac{20{\cdot}25 - 20{\cdot}05}{0{\cdot}1}\right] \\ &= P(Z \le -3) + P(Z \ge 2) \\ &= 1 - P(Z \le 3) + 1 - P(Z \le 2) \\ &= 1 - 0{\cdot}9987 + 1 - 0{\cdot}9772 \\ &= 2 - 1{\cdot}9759 \\ &= 0{\cdot}0241 \end{aligned}$$



$$\frac{0{\cdot}0241}{0{\cdot}0124} = 1{\cdot}9435\ldots \Rightarrow 94{\cdot}35\% \text{ increase}$$

or increase: 0·0241 – 0·0124 = 0·0117

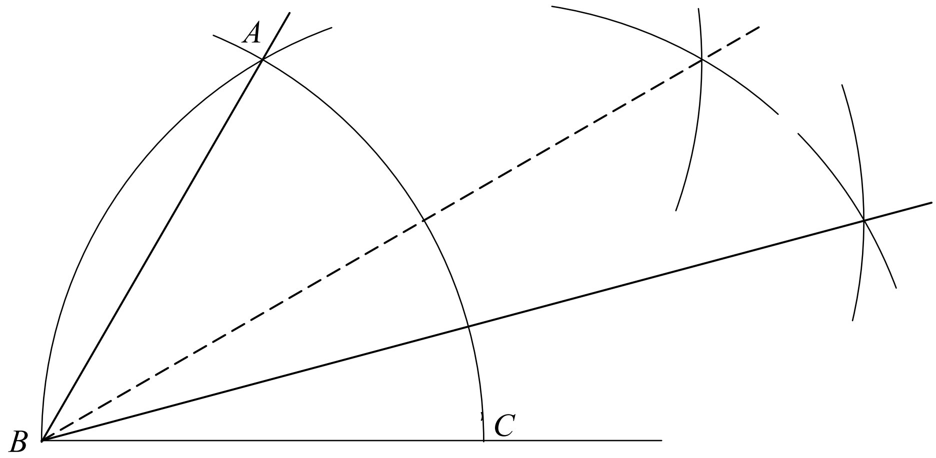
% Increase: $\left(\dfrac{0{\cdot}0117}{0{\cdot}0124}\right)100 = 94{\cdot}35\%$

Question 6 **(25 marks)**

Answer **either** 6A **or** 6B.

Question 6A

(a) **(i)** Given the points B and C below, construct, without using a protractor or setsquare, a point A such that $|\angle ABC| = 60°$.

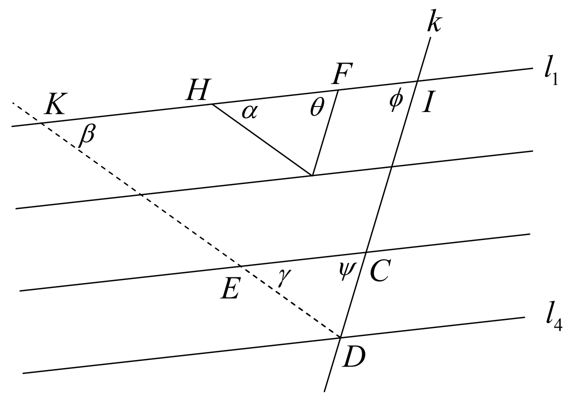


(ii) Hence construct, on the same diagram above, and using a compass and straight edge only, an angle of 15°.

Bisect 60° to get 30°; bisect again to get 15° (as shown above) OR Construct a right angle and use it to construct 45° and combine with 60° to get 15°.

(b) In the diagram, $l_1, l_2, l_3,$ and l_4 are parallel lines that make intercepts of equal length on the transversal k. FG is parallel to k, and HG is parallel to ED.

Prove that the triangles ΔCDE and ΔFGH are congruent.



$\|CD\| = \|IJ\|$	(given)
$= \|FG\|$	(opposite sides of parallelogram
$\theta = \phi = \psi$	(corresponding angles)
$\alpha = \beta = \gamma$	(corresponding angles)
$\Rightarrow \|\angle HGF\| = \|\angle EDC\|$	
$\therefore \Delta CDE \equiv \Delta FGH$	(ASA)

OR

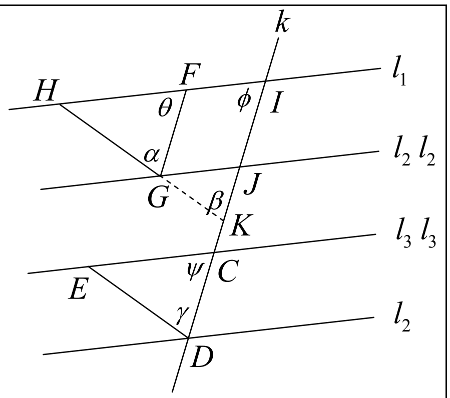
$|CD| = |IJ|$ (given)

$= |FG|$ (opposite sides of parallelogram

$\theta = \phi = \psi$ (corresponding angles)

$\alpha = \beta = \gamma$ (corresponding angles)

$\therefore \Delta CDE \equiv \Delta FGH$ (ASA)

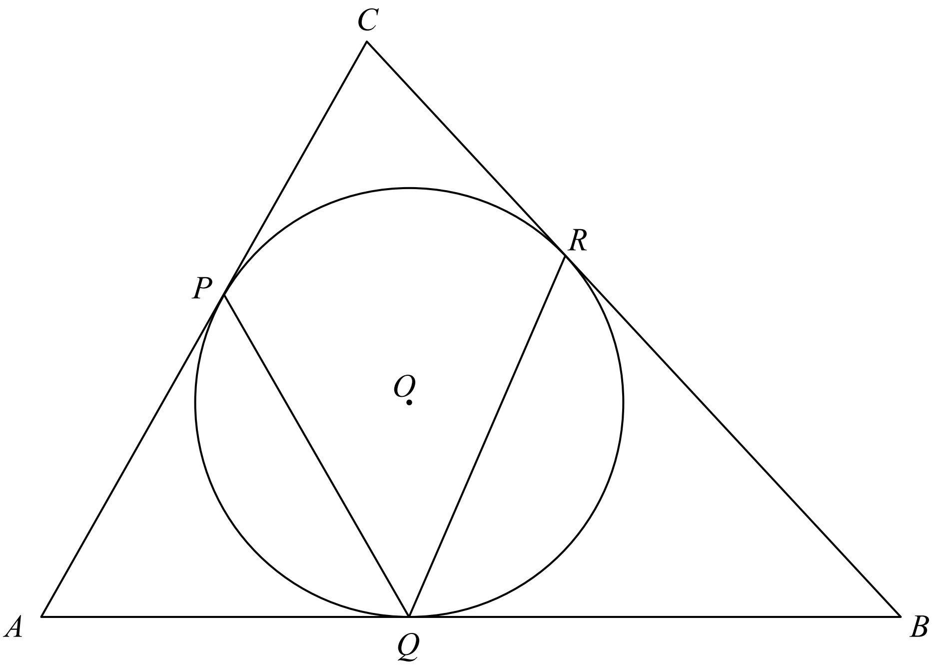


OR

Question 6B

The incircle of the triangle ABC has centre O and touches the sides at P, Q and R, as shown.

Prove that $|\angle PQR| = \frac{1}{2}\left(|\angle CAB| + |\angle CBA|\right)$.



$|\angle OQA| = |\angle OPA| = 90°$ (radius $\perp$ tangent)

$\therefore$ O, Q, A, P are concyclic.

$|\angle OQP| = |\angle OAP|$ (standing on same arc OP)

$= \frac{1}{2}|\angle PAQ|$ (since [AO is the bisector of $\angle PAQ$)

Similarly, $|\angle OQR| = \frac{1}{2}|\angle QBR|$

Adding these two gives the required result.

OR

$\lvert\angle OPC\rvert = \lvert\angle ORC\rvert = 90°$	(radius $\perp$ tangent)
$\therefore \lvert\angle PBR\rvert = 180° - \lvert\angle POR\rvert$	(angles in any quadrilateral add up to 360°)
But $\lvert\angle PBR\rvert = 180° - (\lvert\angle CAB\rvert + \lvert\angle CBA\rvert)$	(angles in a triangle)
So $\lvert\angle POR\rvert = \lvert\angle CAB\rvert + \lvert\angle CBA\rvert$	
But $\lvert\angle PQR\rvert = \frac{1}{2}\lvert\angle POR\rvert$	
So $\lvert\angle PQR\rvert = \frac{1}{2}(\lvert\angle CAB\rvert + \lvert\angle CBA\rvert)$	

OR

Let $OA \cap PQ = \{D\}$	
$\lvert OP\rvert = \lvert OQ\rvert \Rightarrow \lvert AP\rvert = \lvert AQ\rvert$	(Pythagoras)
$\lvert\angle PAD\rvert = \lvert\angle QAD\rvert$	(bisector)
$\therefore \Delta PDA \equiv \Delta QDA$	(S.A.S.)
$\therefore \lvert\angle PDA\rvert = \lvert\angle QDA\rvert = 90°$	
$\lvert\angle DAQ\rvert = 90° - \lvert\angle DQA\rvert$	
$\qquad = \lvert\angle OQD\rvert$	
$\therefore \lvert\angle PAQ\rvert = 2\lvert\angle OQD\rvert$	
Similarly, $\lvert\angle RBQ\rvert = 2\lvert OQR\rvert$	
Adding these two gives the required result.	

OR

Let $OA \cap PQ = \{D\}$	
$\lvert OP\rvert = \lvert OQ\rvert \Rightarrow \lvert AP\rvert = \lvert AQ\rvert$	(Pythagoras)
$\lvert\angle APQ\rvert = \lvert\angle AQP\rvert$	(isosceles triangle theorem)
Similarly, $\lvert\angle RQB\rvert = \lvert\angle RBQ\rvert$	
$\lvert\angle AQP\rvert + \lvert\angle PQR\rvert + \lvert\angle RQB\rvert = 180^0$	
$\lvert\angle PQR\rvert = 180^0 - \lvert\angle AQP\rvert - \lvert\angle RQB\rvert$	
$\lvert\angle CAB\rvert = 180^0 - 2\lvert\angle AQP\rvert$	
$\lvert\angle CBA\rvert = 180^0 - 2\lvert\angle RQB\rvert$	
$\Rightarrow \lvert\angle CAB\rvert + \lvert\angle CBA\rvert = 360^0 - 2[\lvert\angle AQP\rvert + \lvert\angle RQB\rvert]$	
$\Rightarrow \frac{1}{2}[\lvert\angle CAB\rvert + \lvert\angle CBA\rvert] = 180^0 - \lvert\angle AQP\rvert - \lvert\angle RQB\rvert = \lvert\angle PQR\rvert$	

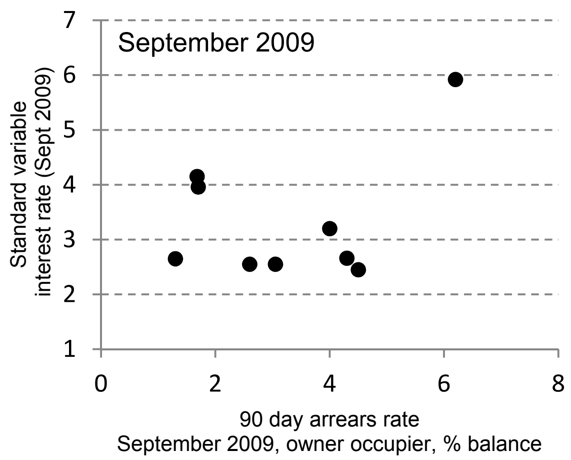
Section B	**Contexts and Applications**	**150 marks**

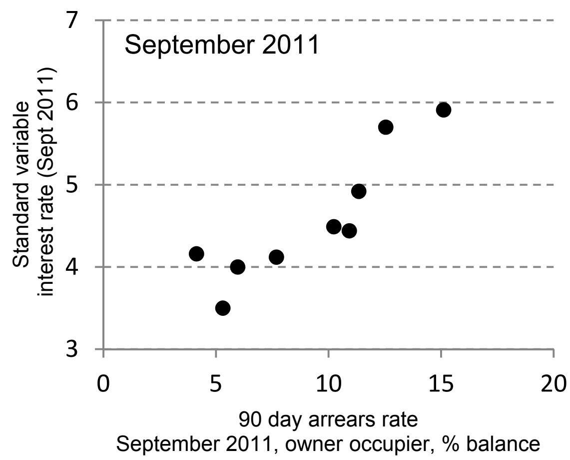
Answer Question 7 and Question 8.

Question 7 **(75 marks)**

To buy a home, people usually take out loans called *mortgages*. If one of the repayments is not made on time, the mortgage is said to be *in arrears*. One way of considering how much difficulty the borrowers in a country are having with their mortgages is to look at the percentage of all mortgages that are in arrears for 90 days or more. For the rest of this question, the term *in arrears* means in arrears for 90 days or more.

The two charts below are from a report about mortgages in Ireland. The charts are intended to illustrate the connection, if any, between the percentage of mortgages that are in arrears and the interest rates being charged for mortgages. Each dot on the charts represents a group of people paying a particular interest rate to a particular lender. The arrears rate is the percentage in arrears.





(Source: Goggin et al.*Variable Mortgage Rate Pricing in Ireland*, Central Bank of Ireland, 2012)

(a) Paying close attention to the scales on the charts, what can you say about the change from September 2009 to September 2011 with regard to:

(i) the arrears rates?

They've gone up a lot – they were mostly between 1 and 5 in 2009, and mostly between 5 and 15 in 2011.

(ii) the rates of interest being paid?

They've gone up a lot too – they were mostly between 2·3 and 4·1% in 2009, and mostly between 4 and 6% in 2011.

(iii) the relationship between the arrears rate and the interest rate?

There appears to be a stronger relationship 2011 than in 2009.

(b) What additional information would you need before you could estimate the median interest rate being paid by mortgage holders in September 2011?

You would need to know how many mortgage holders are represented by each point on the relevant diagram.

(c) Regarding the relationship between the arrears rate and the interest rate for September 2011, the authors of the report state: "The direction of causality … is important" and they go on to discuss this.

Explain what is meant by the "direction of causality" in this context.

It is a question of whether higher arrears rates cause interest rates to go up, or whether higher interest rates cause arrears rates to go up, (assuming there is a causal relationship at all).

(d) A property is said to be in "negative equity" if the person owes more on the mortgage than the property is worth. A report about mortgaged properties in Ireland in December 2010 has the following information:

- Of the 475 136 properties examined, 145 414 of them were in negative equity.
- Of the ones in negative equity, 11 644 were in arrears.
- There were 317 355 properties that were neither in arrears nor in negative equity.

(i) What is the probability that a property selected at random (from all those examined) will be in negative equity?
Give your answer correct to two decimal places.

$$\frac{145414}{475136} = 0{\cdot}30604711 = 0{\cdot}31 \text{ (to two decimal places)}$$

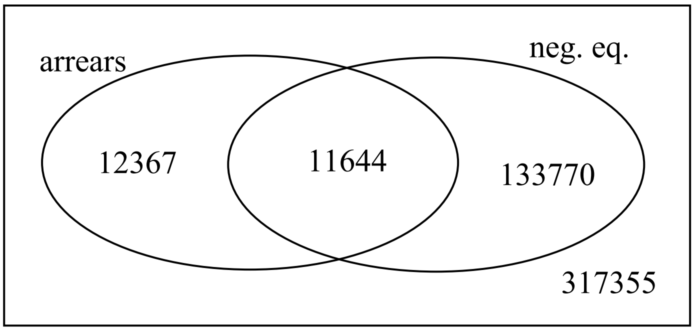
(ii) What is the probability that a property selected at random from all those in negative equity will also be in arrears?
Give your answer correct to two decimal places.

$$\frac{11644}{145414} = 0{\cdot}08007482 = 0{\cdot}08 \text{ (to two decimal places)}$$

(iii) Find the probability that a property selected at random from all those in arrears will also be in negative equity.
Give your answer correct to two decimal places.

	arrears	¬arrears	total
neg. eq.	**11644**	133770	**145414**
¬neg. eq.	12367	**317355**	329722
total	24011	451125	**475136**

$$\frac{11644}{24011} = 0{\cdot}4849 = 0{\cdot}48 \text{ (to two decimal places)}$$



OR

$$P(A|N) = \frac{P(A \cap N)}{P(N)} \Rightarrow 0{\cdot}08007 = \frac{P(A \cap N)}{0{\cdot}30604} \Rightarrow P(A \cap N) = 0{\cdot}0245$$

$$\text{But } P(A) = \frac{24011}{475136} = 0{\cdot}05053$$

$$P(N|A) = \frac{P(N \cap A)}{P(A)} = \frac{0{\cdot}0245}{0{\cdot}05053} = 0{\cdot}4848 = 0{\cdot}48 \text{ (to two decimal places)}$$

(e) The study described in part **(d)** was so large that it can be assumed to represent the population. Suppose that, in early 2012, researchers want to know whether the proportion of properties in negative equity has changed. They analyse 2000 randomly selected properties with mortgages. They discover that 552 of them are in negative equity. Use a hypothesis test at the 5% level of significance to decide whether there is sufficient evidence to conclude that the situation has changed since December 2010.

Be sure to state the null hypothesis clearly, and to state the conclusion clearly.

Null hypothesis: proportion in negative equity unchanged: $p = 0{\cdot}31$.
Alternative hypothesis: it has changed: $p \neq 0{\cdot}31$.

95% margin of error for samples of size 2000 is $\frac{1}{\sqrt{2000}} \approx 0{\cdot}0224$

So, reject null hypothesis if observed proportion lies outside $0{\cdot}31 \pm 0{\cdot}0224$.

Observed proportion = $\frac{552}{2000} \approx 0{\cdot}276$.

$0{\cdot}276 \notin [0{\cdot}2876, 0{\cdot}3224]$

Outside margin of error, so reject null hypothesis.

The proportion in negative equity has changed.

OR

Null hypothesis: proportion in negative equity unchanged: $p = 0{\cdot}31$.

95% margin of error for samples of size 2000 is $\frac{1}{\sqrt{2000}} \approx 0{\cdot}0224$

Observed proportion = $\frac{552}{2000} \approx 0{\cdot}276$.

$\therefore$ the 95% confidence interval for the population proportion is:

$0{\cdot}276 - 0{\cdot}0224 < p < 0{\cdot}276 + 0{\cdot}0244$

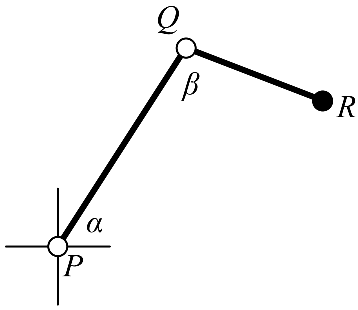
$0{\cdot}2536 < p < 0{\cdot}2984$

$0{\cdot}31$ outside this range.

Therefore reject null hypothesis. Proportion in negative equity has changed.

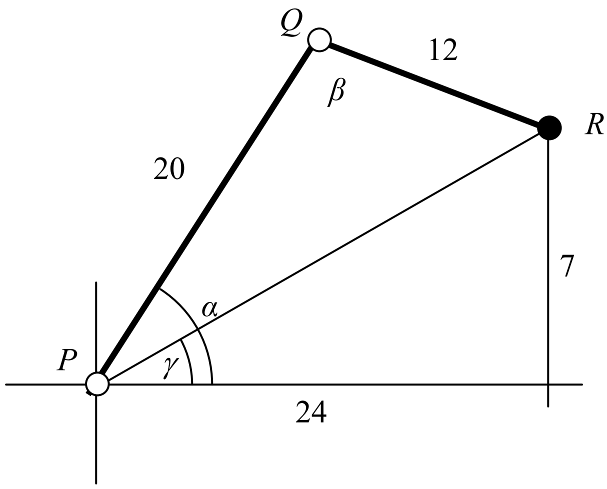
Question 8 **(75 marks)**

The diagram is a representation of a robotic arm that can move in a vertical plane. The point P is fixed, and so are the lengths of the two segments of the arm. The controller can vary the angles α and β from 0° to 180°.



(a) Given that $|PQ| = 20$ cm and $|QR| = 12$ cm, determine the values of the angles α and β so as to locate R, the tip of the arm, at a point that is 24 cm to the right of P, and 7 cm higher than P. Give your answers correct to the nearest degree.

$$|PR|^2 = 7^2 + 24^2$$
$$|PR| = 25$$



$$25^2 = 20^2 + 12^2 - 2(20)(12)\cos\beta$$
$$\cos\beta = -0{\cdot}16875$$
$$\beta \approx 100°$$

$$12^2 = 25^2 + 20^2 - 2(25)(20)\cos(\alpha - \gamma)$$
$$\cos(\alpha - \gamma) = 0{\cdot}881$$
$$\alpha - \gamma \approx 28{\cdot}237°$$

$$\tan\gamma = \frac{7}{24}$$
$$\gamma \approx 16{\cdot}260°$$

$$\therefore \alpha \approx 44°$$

(b) In setting the arm to the position described in part **(a)**, which will cause the greater error in the location of R: an error of $1°$ in the value of α or an error of $1°$ in the value of β?

Justify your answer. You may assume that if a point moves along a circle through a small angle, then its distance from its starting point is equal to the length of the arc travelled.

Ans: α

Reason: $1°$ error in α causes R to move along an arc of radius 25.
$1°$ error in β causes R to move along an arc of radius 12.

So, since $l = r\theta$, and θ is the same in each case, the point moves further in the first case.

(c) The answer to part **(b)** above depends on the particular position of the arm. That is, in certain positions, the location of R is more sensitive to small errors in α than to small errors in β, while in other positions, the reverse is true. Describe, with justification, the conditions under which each of these two situations arises.

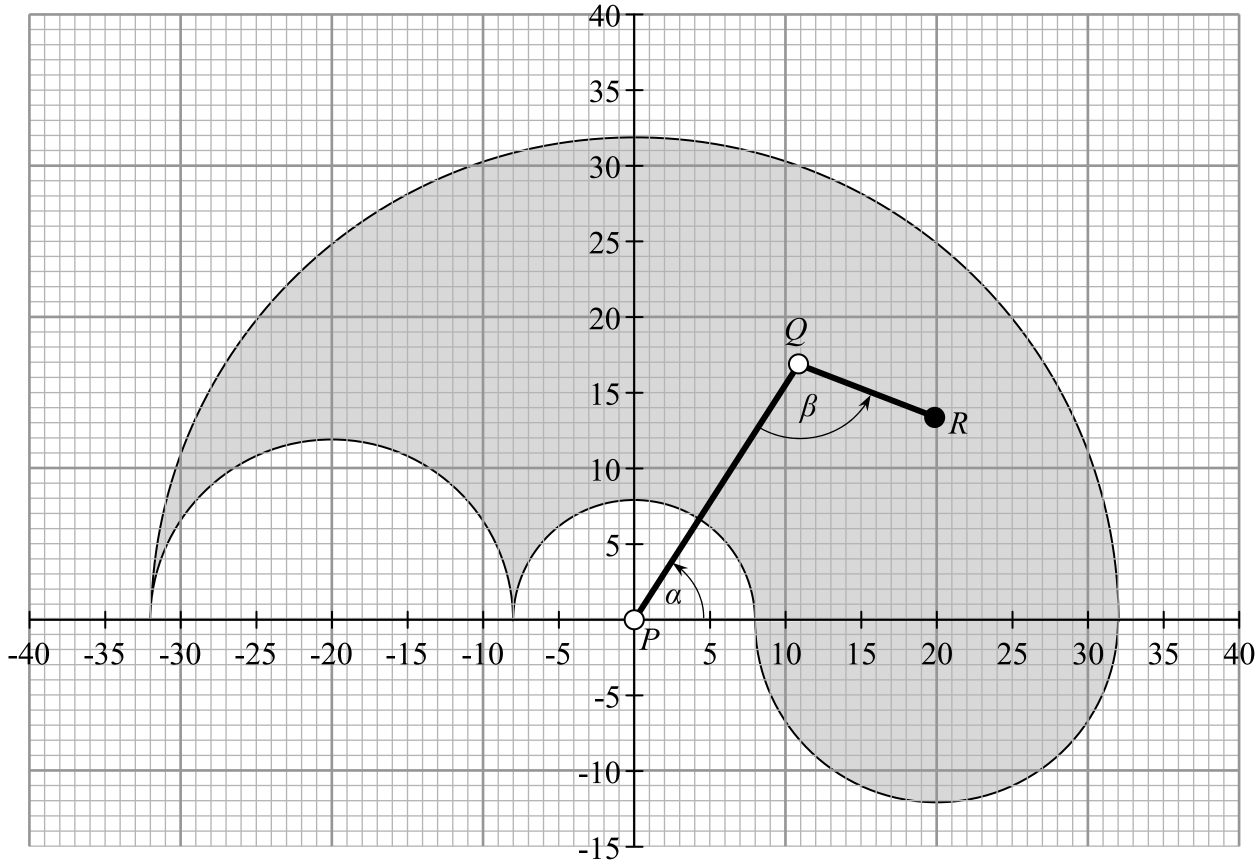
More sensitive to errors in α when $|PR| > 12$

More sensitive to errors in β when $|PR| < 12$

The condition $|PR| > 12$
is true whenever
$\beta > \cos^{-1}\left(\frac{5}{6}\right) \approx 33{\cdot}6°$

(Borderline case is when ΔPQR is isosceles with $|QR| = |RP|$.)

(d) Illustrate the set of all possible locations of the point R on the coordinate diagram below. Take P as the origin and take each unit in the diagram to represent a centimetre in reality. Note that α and β can vary only from 0° to 180°.



Marking Scheme – Paper 2

Structure of the marking scheme

Candidate responses are marked according to different scales, depending on the types of response anticipated. Scales labelled A divide candidate responses into two categories (correct and incorrect). Scales labelled B divide responses into three categories (correct, partially correct, and incorrect), and so on. The scales and the marks that they generate are summarised in this table:

Scale label	A	B	C	D	E
No of categories	2	3	4	5	6
5 mark scale		0, 3, 5	0, 3, 4, 5		
10 mark scale		0, 5, 10	0, 4, 8, 10		
15 mark scale					
20 mark scale			0, 7, 18, 20	0, 7, 10, 18, 20	
25 mark scale				0, 15, 20, 22, 25	0, 5, 10, 15, 20, 25

A general descriptor of each point on each scale is given below. More specific directions in relation to interpreting the scales in the context of each question are given in the scheme, where necessary.

Marking scales – level descriptors

A-scales (two categories)
- incorrect response (no credit)
- correct response (full credit)

B-scales (three categories)
- response of no substantial merit (no credit)
- partially correct response (partial credit)
- correct response (full credit)

C-scales (four categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

D-scales (five categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response about half-right (middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

E-scales (six categories)

- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response almost half-right (lower middle partial credit)
- response more than half-right (upper middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

In certain cases, typically involving incorrect rounding or omission of units, a mark that is one mark below the full-credit mark may also be awarded. Such cases are flagged with an asterisk. Thus, for example, *scale 10C** indicates that 9 marks may be awarded.

Summary of mark allocations and scales to be applied

Section A

Question 1

(a) 5B, 5B, 5B
(b) 10C

Question 2

(a) 5B, 5B
(b) 5C
(c) 5B
(d) 5B

Question 3

25D

Question 4

(a) 5C
(b)(i) 10C*
(b)(ii) 10C*

Question 5

(a) 20D
(b) 5C

Question 6A

(a)(i) 10C
(a)(ii)5C
(b) 10C

Question 6B

25E

Section B

Question 7

(a)(i) 5B
(a)(ii) 5B
(a)(iii) 10B
(b) 5B
(c) 5C
(d)(i) 10C*
(d)(ii) 5B*
(d)(iii) 10C*
(e) 20D

Question 8

(a)$|PR|$ 10B
(a)β 20C*
(a)α 25D*
(b) 5C
(c) 5C
(d) 10C

Detailed marking notes

Section A

Question 1

(a) Scale 5B, 5B, 5B (0, 3, 5)

Partial Credit:
- Incomplete statement of method (with some merit)

(b) Scale 10C (0, 4, 8, 10).

Low Partial Credit:
- Any reasonable first step.

High Partial Credit:
- Correct method applied with some error(s)
- Correct method but vertices not taken in correct order (i.e *ABDC* or equivalent)
- Correct method completed but with no conclusion.

Question 2

(a) Scale 5B, 5B (0, 3, 5)

Partial Credit:
- Centre or radius found
- Incomplete statement of method (with some merit) e.g., completing square

(b) Scale 5C (0, 3, 4, 5)

Low Partial Credit:
- Formula for distance between two centres with some substitution
- Difference between radii found or implied

High Partial Credit:
- No conclusion or incorrect conclusion

(c) Scale 5B (0, 3, 5)

Partial Credit:
- (4, 7) substituted into one circle
- Subtracts both equations and stops
- (4, 7) substituted into both circles with an error
- Finds common tangent and shows that (4, 7) is on it, without showing that it is also on one of the circles

(d) Scale 5B (0, 3, 5)

Partial Credit:
- Some relevant slope found
- (4, 7) inserted into equation of line formula but without slope found
- Equation of line but (4, 7) incorrectly inserted

Question 3

Scale 25D (0, 15, 20, 22, 25)

Low Partial Credit:

- Any reasonable first step, such as:
  - radius / diameter indicated to one of the points of contact
  - intercepts of either tangent on axes indicated
  - (1, 1) and/or (k, k) on diagram with no further work of merit
  - $|g| = |f|$ or g = $\pm f$
  - centre ($-g$, $-g$) or equivalent
- Perpendicular distance of centre to either tangent indicated

Mid Partial Credit:

- Centre (r, r)
- Equation connecting r and k (i.e. work towards $2r - 1 = k$)
- Writes equation of circle $(x-r)^2 + (y-r)^2 = r^2$
- Equation $x^2 + y^2 + 2gx + 2gy + g^2 = 0$ or equivalent
- $y = x$ and further work of merit
- States $g = f$ and $g^2 = f^2 = c$
- Perpendicular distance ($-g$, $-g$) to tangent
- Substantive work at finding equation of a relevant angle-bisector, other than $y = x$

High Partial Credit:

- $r = 2+\sqrt{2}$ or equivalent but fails to finish

Question 4

(a) Scale 5C (0, 3, 4, 5)

Low Partial Credit:
- One or both ‘given’ assumptions stated or implied

High Partial Credit:
- Either ‘independence’ or ‘probability of success the same each time’ stated.

(b)(i) Scale 10C* (0, 4, 8, [9] ,10)

Low Partial Credit:
- Any first step e.g. reference to 0·4 or equivalent

High Partial Credit:
- Correct expression
- Answer with one error in components

Note: Rounding incomplete: 9 marks

(b)(ii) Scale 10C* (0, 4, 8, [9], 10)

Low Partial Credit:
- Reference to 0·6 or equivalent for fifth shot

High Partial Credit:
- Correct expression
- Answer with one error in components

Note: Rounding incomplete: 9 marks

Question 5

(a) Scale 20D (0, 7, 10, 18, 20)

Low Partial Credit:
- Any relevant step
- Some relevant diagram

Mid Partial Credit:
- Reference to 2·5
- $P(Z > 2{\cdot}5) = 0{\cdot}0062$ and stops

High Partial Credit:
- $P(|Z| > 2{\cdot}5) = 0{\cdot}0124$
- Correct method with some error

(b) Scale 5C (0, 3, 4, 5)

Low Partial Credit:
- Any relevant step
- Some relevant diagram
- One case taken only

High Partial Credit:
- Probability of both situations calculated but fails to complete fully

Question 6A

(a)(i) Scale 10C (0, 4, 8, 10)

Low Partial Credit:
- Any correct step

High Partial Credit:
- Correct method but outside the tolerance of 2°

(a)(ii) Scale 5C (0, 3, 4, 5)

Low Partial Credit:
- Any correct step

High Partial Credit:
- Correct method but outside the tolerance of 2°

(b) Scale 10C (0, 4, 8, 10)

Low Partial Credit:
- Any correct step e.g.,
    - identifies two equal sides
    - identifies two equal angles
    - extends *DE* to intersect l_1

High Partial Credit:
- Proof with correct steps but without justification of steps
- One error in establishing congruence

Question 6B

Scale 25E (0, 5, 10, 15, 20, 25)

Low Partial Credit:
- Any correct statement

Lower Middle Partial Credit:
- Some substantive work towards proof e.g. at least one full step complete
- Two distinct relevant statements

Upper Middle Partial Credit:
- Substantive proof with two critical steps missing

High Partial Credit:
- Correct proof but critical step missing
- Correct proof without justification of steps

Question 7

(a)(i) Scale 5B (0, 3, 5)

Partial Credit
- Incomplete statement e.g., "it has changed".

(a)(ii) Scale 5B (0, 3, 5)

Partial Credit
- Incomplete or partly correct statement

(a)(iii) Scale 10B (0, 5, 10)

Partial Credit
- Incomplete or partly correct statement, e.g.,
    - "They have changed"
    - "Closer to being a line"
    - Reference to positive

(b) Scale 5B (0, 3, 5)

Partial Credit
- Some reference to the middle household

Note: accept (for full credit) reference to needing information about mortgage holders who are not on standard variable rates.

(c) Scale 5C (0, 3, 4, 5)

Low Partial Credit
- Both may be caused by something else
- General statement regarding correlation not implying causality – no context

High Partial Credit
- No reference to reverse situation, (e.g.: "It's about whether high interest rates cause high arrears rates or not.")
- Correct interpretation of concept, but not contextualised, (e.g. "It's a question of which variable causes which.")

(d)(i) Scale 10C* (0, 4, 8, [9], 10)

Low Partial Credit

- Uses a relevant number
- Writes $\frac{\#E}{\#S}$ or equivalent.
- Identifies "number of outcomes of interest = ..." or "total number of outcomes = ...".

High Partial Credit

- Answer in the form of a fraction

(d)(ii) Scale 5B* (0, 3, [4], 5)

Partial Credit

- Uses a relevant number
- Writes $\frac{\#E}{\#S}$ or equivalent.
- Identifies "number of outcomes of interest = ..." or "total number of outcomes = ...".
- Answer in form of a fraction

(d)(iii) Scale 10C* (0, 4, 8, [9], 10)

Low Partial Credit

- Attempt to combine (i) and (ii) for part (iii)
- Calculates total arrears and stops

High Partial Credit

- Answer as a fraction
- $\frac{11644}{12367} = 0{\cdot}9415 = 0{\cdot}94$

(e) Scale 20D (0, 7, 10, 18, 20)

Low Partial Credit

- One relevant step e.g. null hypothesis stated only
- Margin of error or observed proportion and does not continue

Mid Partial Credit:

- Substantive work with one or more critical omissions
- Margin of error and observed proportion found but fails to continue

High Partial Credit

- Failure to state null hypothesis correctly and/or failure to contextualise answer (e.g., stops at "Reject the null hypothesis".

Question 8

(a) $|PR|$ Scale 10B (0, 5, 10)

Partial Credit
- Some use of Pythagoras

(a) β Scale 20C* (0, 7, 18, [19], 20)

Low Partial Credit
- Cosine Rule with some substitution

High Partial Credit
- Cos β calculated

(a) α Scale 25D* (0, 15, 20, 22, [24], 25)

Low Partial Credit
- Some work towards solving required angle with *sine* or *cosine* rule
- Tan $\gamma = \frac{7}{24}$ with no work towards $\alpha - \gamma$

Middle Partial Credit
- Cos $\alpha - \gamma$ found

High Partial Credit
- $\alpha - \gamma$ and γ calculated but α not evaluated

(b) Scale 5C (0, 3, 4, 5)

Low Partial Credit
- Effort at working out values for angles
- Correct answer without justification

High Partial Credit
- Correct answer without being fully justified

(c) Scale 5C (0, 3, 4, 5)

Low Partial Credit
- Some reference to distance between *P* and *R*
- Treats as percentage error in angles, rather than absolute error in location. e.g. "If α is smaller than β, then a 1° error in α is a bigger percentage error than a 1° error in β."

High Partial Credit
- One situation only dealt with correctly
- Clearly understands concept that the radius of the rotation is the determining factor, but makes error(s) in explanation (e.g. mixes up distances involved).

(d) Scale 10C (0, 4, 8, 10)

Low Partial Credit
- Any relevant semi circle sketched or implied

High Partial Credit
- Any correct semi circle inserted in addition to semicircle centre *P* with radius 32.

Marcanna Breise as ucht Freagairt trí Ghaeilge

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronnadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d'iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú **síos**.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéir ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. 198 marc × 5% = 9·9 ⇒ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, ríomhtar an bónas de réir na foirmle [300 – bunmharc] × 15%, agus an marc bónais sin a shlánú **síos**. In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

Bunmharc	Marc Bónais
226	11
227 – 233	10
234 – 240	9
241 – 246	8
247 – 253	7
254 – 260	6
261 – 266	5
267 – 273	4
274 – 280	3
281 – 286	2
287 – 293	1
294 – 300	0