LEAVING CERTIFICATE EXAMINATION, 2008

MATHEMATICS — HIGHER LEVEL

PAPER 1 (300 marks)

FRIDAY, 6 JUNE – MORNING, 9:30 to 12:00

Attempt SIX QUESTIONS (50 marks each).

WARNING: Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.
1. (a) Simplify fully \( \frac{x^2 + 4}{x^2 - 4} - \frac{x}{x + 2} \).

(b) Given that one of the roots is an integer, solve the equation 
\[ 6x^3 - 29x^2 + 36x - 9 = 0. \]

(c) Two of the roots of the equation \( ax^3 + bx^2 + cx + d = 0 \) are \( p \) and \( -p \). Show that \( bc = ad \).

2. (a) Express \( x^2 + 10x + 32 \) in the form \( (x + a)^2 + b \).

(b) \( \alpha \) and \( \beta \) are the roots of the equation \( x^2 - 7x + 1 = 0 \).

(i) Find the value of \( \alpha^2 + \beta^2 \).

(ii) Find the value of \( \frac{1}{\alpha^3} + \frac{1}{\beta^3} \).

(c) Show that if \( a \) and \( b \) are non-zero real numbers, then the value of \( \frac{a}{b} + \frac{b}{a} \) can never lie between \(-2\) and \(2\).

Hint: consider the case where \( a \) and \( b \) have the same sign separately from the case where \( a \) and \( b \) have opposite sign.
3. (a) Let $A$ be the matrix \[
\begin{pmatrix}
3 & 5 \\
1 & 2 
\end{pmatrix}
\]
Find the matrix $B$, such that $AB = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix}$.

(b) (i) Let $z = \frac{5}{2+i} - 1$, where $i^2 = -1$.
Express $z$ in the form $a + bi$ and plot it on an Argand diagram.

(ii) Use De Moivre’s theorem to evaluate $z^6$.

(c) Prove, by induction, that
\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad \text{for } n \in \mathbb{N}.
\]

4. (a) $2 + \frac{2}{3} + \frac{2}{9} + \ldots$ is a geometric series.
Find the sum to infinity of the series.

(b) Given that $u_n = 2\left(-\frac{1}{2}\right)^n - 2$ for all $n \in \mathbb{N}$,

(i) write down $u_{n+1}$ and $u_{n+2}$

(ii) show that $2u_{n+2} - u_{n+1} - u_n = 0$.

(c) (i) Write down an expression in $n$ for the sum $1 + 2 + 3 + \ldots + n$
and an expression in $n$ for the sum $1^2 + 2^2 + 3^2 + \ldots + n^2$.

(ii) Find, in terms of $n$, the sum $\sum_{r=1}^{n} \left(6r^2 + 2r + 5 + 2r\right)$.  

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5. (a) Find the range of values of $x$ that satisfy the inequality $x^2 - 3x - 10 \leq 0$.

(b) (i) Solve the equation $2x^2 = 8^{2x+9}$.

(ii) Solve the equation $\log_e(2x + 3) + \log_e(x - 2) = 2 \log_e(x + 4)$.

(c) Show that there are no natural numbers $n$ and $r$ for which

$$\binom{n}{r} \binom{n}{r-1} \quad \text{and} \quad \binom{n}{r+1}$$

are consecutive terms in a geometric sequence.

6. (a) Differentiate $\sqrt{x^3}$ with respect to $x$.

(b) Let $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Show that $\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}$.

(c) The function $f(x) = 2x^3 + 3x^2 + bx + c$ has a local maximum at $x = -2$.

(i) Find the value of $b$.

(ii) Find the range of values of $c$ for which $f(x) = 0$ has three distinct real roots.
7. (a) Differentiate $2x + \sin 2x$ with respect to $x$.

(b) The equation of a curve is $5x^2 + 5y^2 + 6xy = 16$.

(i) Find $\frac{dy}{dx}$ in terms of $x$ and $y$.

(ii) $(1, 1)$ and $(2, -2)$ are two points on the curve. Show that the tangents at these points are perpendicular to each other.

(c) Let $y = \sin^{-1}\left(\frac{x}{\sqrt{1 + x^2}}\right)$.

Find $\frac{dy}{dx}$ and express it in the form $\frac{a}{a + x^b}$, where $a, b \in \mathbb{N}$.

8. (a) Find $\int (2x + \cos 3x) \, dx$.

(b) Evaluate (i) $\int_0^1 3x^2 e^{-x^3} \, dx$ (ii) $\int_2^4 \frac{2x^3}{x^2 - 1} \, dx$.

(c) The diagram shows the curve $y = 4 - x^2$ and the line $2x + y - 1 = 0$.

Calculate the area of the shaded region enclosed by the curve and the line.