Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE 2009

MARKING SCHEME

MATHEMATICS

HIGHER LEVEL
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>GENERAL GUIDELINES FOR EXAMINERS – PAPER 1</td>
<td>4</td>
</tr>
<tr>
<td>QUESTION 1</td>
<td>5</td>
</tr>
<tr>
<td>QUESTION 2</td>
<td>12</td>
</tr>
<tr>
<td>QUESTION 3</td>
<td>16</td>
</tr>
<tr>
<td>QUESTION 4</td>
<td>19</td>
</tr>
<tr>
<td>QUESTION 5</td>
<td>23</td>
</tr>
<tr>
<td>QUESTION 6</td>
<td>26</td>
</tr>
<tr>
<td>QUESTION 7</td>
<td>30</td>
</tr>
<tr>
<td>QUESTION 8</td>
<td>33</td>
</tr>
<tr>
<td>GENERAL GUIDELINES FOR EXAMINERS – PAPER 2</td>
<td>37</td>
</tr>
<tr>
<td>QUESTION 1</td>
<td>38</td>
</tr>
<tr>
<td>QUESTION 2</td>
<td>43</td>
</tr>
<tr>
<td>QUESTION 3</td>
<td>46</td>
</tr>
<tr>
<td>QUESTION 4</td>
<td>51</td>
</tr>
<tr>
<td>QUESTION 5</td>
<td>54</td>
</tr>
<tr>
<td>QUESTION 6</td>
<td>59</td>
</tr>
<tr>
<td>QUESTION 7</td>
<td>64</td>
</tr>
<tr>
<td>QUESTION 8</td>
<td>68</td>
</tr>
<tr>
<td>QUESTION 9</td>
<td>72</td>
</tr>
<tr>
<td>QUESTION 10</td>
<td>76</td>
</tr>
<tr>
<td>QUESTION 11</td>
<td>79</td>
</tr>
<tr>
<td>MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAEILGE</td>
<td>82</td>
</tr>
</tbody>
</table>
GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates’ work as follows:
   • Blunders - mathematical errors/omissions (-3)
   • Slips - numerical errors (-1)
   • Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   • any correct, relevant step in a part of a question merits at least the attempt mark for that part
   • if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   • a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The same error in the same section of a question is penalised once only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.


QUESTION 1

Part (a) 10 (5, 5) marks  Att (2, 2)
Part (b) 20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)
Part (c) 20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

Part (a) 10 (5, 5) marks  Att (2, 2)

1. (a) Find the value of $\frac{x}{y}$ when $\frac{2x + 3y}{x + 6y} = \frac{4}{5}$.

Cross Multiplication  5 marks  Att 2
Finish  5 marks  Att 2

1 (a)

<table>
<thead>
<tr>
<th>$\frac{2x + 3y}{x + 6y} = \frac{4}{5}$</th>
<th>$\Rightarrow 10x + 15y = 4x + 24y \Rightarrow 6x = 9y$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\therefore \frac{x}{y} = \frac{9}{6} = \frac{3}{2}$</td>
<td>$\therefore \frac{x}{y} = \frac{9}{6} = \frac{3}{2}$.</td>
</tr>
</tbody>
</table>

Blunders (-3)
B1 Incorrect cross multiplication

Slips (-1)
S1 Numerical
S2 $\frac{y}{x}$

OR

Correct Ratio  5 marks  Att 2
Solving  5 marks  Att 2

1 (a)

Let numerator = 4 and denominator = 5 (or 8 & 10 respectively, etc.)

$\Rightarrow (i) : 2x + 3y = 4 \times 2 \Rightarrow 4x + 6y = 8$

$(ii) : x + 6y = 5 \times 1 \Rightarrow x + 6y = 5$

$3x = 3 \Rightarrow x = 1$

(ii): $x + 6y = 5$

$(i) + 6y = 5$

$6y = 4 \Rightarrow y = \frac{4}{6} = \frac{2}{3}$

$\frac{x}{y} = \left(\frac{2}{3}\right) = \frac{3}{2}$

Blunders (-3)
B1 Error in ratio
B2 No $\frac{x}{y}$

Slips (-1)
S1 Numerical
S2 $\frac{y}{x}$
Part (b)  

**20 (5, 5, 5, 5) marks**  

**Att (2, 2, 2, 2)**

(b) Let \( f(x) = x^2 - 7x + 12 \).

(i) Show that if \( f(x+1) \neq 0 \), then \( \frac{f(x)}{f(x+1)} \) simplifies to \( \frac{x-4}{x-2} \).

(ii) Find the range of values of \( x \) for which \( \frac{f(x)}{f(x+1)} > 3 \).

---

(b) (i) \( f(x+1) \)

<table>
<thead>
<tr>
<th>Simplification</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = x^2 - 7x + 12 ) ( \Rightarrow f(x+1) = (x+1)^2 - 7(x+1) + 12. )</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
<tr>
<td>( f(x) = x^2 - 7x + 12 ) ( \Rightarrow f(x+1) = (x-3)(x-4) ) | ( x-4 ) ( x-2 ) | ( x-2 ) ( x-2 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Blunders (-3)**

B1 Expansion \((x+1)^2\) once only  
B2 Incorrect fraction  
B3 Factors

(b) (ii) Quadratic Inequality

<table>
<thead>
<tr>
<th>Range</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{f(x)}{f(x+1)} &gt; 3 ) ( \Rightarrow \frac{x-4}{x-2} &gt; 3 ) ( \Rightarrow \frac{x-4}{x-2} &gt; 3 ) ( \Rightarrow \frac{x-4}{x-2} &gt; 3 ) ( \Rightarrow \frac{x-4}{x-2} &gt; 3 ) ( \Rightarrow \frac{x-4}{x-2} &gt; 3 ) ( \Rightarrow \frac{x-4}{x-2} &gt; 3 ) ( \Rightarrow \frac{x-4}{x-2} &gt; 3 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Multiply across by \((x-2)^2 > 0\)

\( (x-2)(x-4) > 3(x-2)^2 \)
\( x^2 - 6x + 8 > 3(x^2 - 4x + 4) \)
\( x^2 - 6x + 8 > 3x^2 - 12x + 12 \)
\( 0 > 2x^2 - 6x + 4 \)
\( 0 > x^2 - 3x + 2 \)
\( 0 > (x-1)(x-2) \)

Range: \( 1 < x < 2 \)

**Blunders (-3)**

B1 Inequality sign  
B2 Indices  
B3 Expansion of \((x-2)^2\) once only  
B4 Factors  
B5 Roots formula once only
<table>
<thead>
<tr>
<th>Case</th>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
</table>
| (a)          | \( x - 2 > 0 \) (so \( x > 2 \)) | \( \frac{x - 4}{x - 2} > 3 \)  
\( \iff (x - 4) > 3(x - 2) \) since \( x - 2 > 0 \)  
\( \iff x - 4 > 3x - 6 \)  
\( \iff 2 > 2x \)  
\( \iff 1 > x \)  
Not possible when \( x > 2 \) \( \Rightarrow \) no solution from this case. |
| (b)          | \( x - 2 < 0 \) (so \( x < 2 \)) | \( \frac{x - 4}{x - 2} > 3 \)  
\( \iff x - 4 < 3(x - 2) \) since \( x - 2 < 0 \)  
\( \iff x - 4 < 3x - 6 \)  
\( \iff 2 < 2x \)  
\( \iff 1 < x \)  
\( \Rightarrow 1 < x < 2 \) |
(b) (ii) case \((x - 2) > 0\)  
\[
\frac{x - 4}{x - 2} > 3 \quad \Rightarrow \quad \frac{x - 4}{x - 2} - 3 > 0
\]
\[
\frac{(x - 4) - 3(x - 2)}{(x - 2)} > 0
\]
\[
\frac{x - 4 - 3x + 6}{(x - 2)} > 0
\]
\[
\frac{-2x + 2}{x - 2} > 0
\]
So, need numerator and denominator to have same sign.

- **case (a):** \((x - 2) > 0\) and \(-2x + 2 > 0\)
  \[
x > 2 \quad 2 > 2x \quad 1 > x
\]
  Not possible \(\Rightarrow\) no solution from this case.

- **case (b):** \((x - 2) < 0\) and \(-2x + 2 < 0\)
  \[
x < 2 \quad 2 < 2x \quad 1 < x
\]
  \(\Rightarrow\) \(1 < x < 2\)

**Blunders (-3)**
- B1 Inequality sign
- B2 Deduction of value
- B3 Range not stated
- B4 Incorrect range

**Slips (-1)**
- S1 Numerical
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(c) Given that \(x - c + 1\) is a factor of \(x^2 - 5x + 5cx - 6b^2\), express \(c\) in terms of \(b\).

\[
\begin{align*}
\text{Division} & \quad 5 \text{ marks} & \text{Att} \ 2 \\
\text{Remainder} = 0 & \quad 5 \text{ marks} & \text{Att} \ 2 \\
\text{Quadratic in } b \text{ and } c & \quad 5 \text{ marks} & \text{Att} \ 2 \\
\text{Values of } c & \quad 5 \text{ marks} & \text{Att} \ 2 \\
\end{align*}
\]

1 (c)

\[
\frac{x + (6 + 6c)}{x - c + 1} x^2 - 5x + 5cx - 6b^2
\]

\[
\begin{align*}
&= x^2 + x - cx \\
&= x(6 + 6c) - 6b^2 \\
&= x(-6 + 6c) - c(-6 + 6c) + (-6 + 6c) \\
&= -6b^2 + c(-6 + 6c) - (-6 + 6c) \\
\end{align*}
\]

\[
\therefore \quad -6b^2 - 6c + 6c^2 + 6 - 6c = 0 \\
c^2 - 2c + 1 = b^2. \\
(c - 1)^2 = b^2 \Rightarrow c - 1 = \pm b \Rightarrow c = 1 \pm b.
\]

**Blunders (−3)**
B1 Indices
B2 Not like to like when equation coefficients
B3 Only one value of \(c\) given
B4 Factors

**Slips (−1)**
S1 Not changing sign when subtracting

**Attempts**
A1 Any effort at division
Other linear factor 5 marks
Equating coefficients 5 marks
Quadratic in b and c 5 marks
Values of c 5 marks

1 (c)

\[ f(x) = x^2 - 5x + 5cx - 6b^2 = (x-c+1) \left( x - \frac{6b^2}{-c+1} \right) \]

\[ (x+1-c) \left( x - \frac{6b^2}{1-c} \right) = x^2 - cx + \frac{6b^2x}{1-c} + \frac{6b^2c}{1-c} - \frac{6b^2}{1-c} \]

\[ = x^2 - x \left( c-1 + \frac{6b^2}{1-c} \right) + \frac{6b^2c - 6b^2}{1-c} \]

Equating Coefficients of x:

\[ 5 - 5c = c - 1 + \frac{6b^2}{1-c} \]

\[ 6 - 6c = \frac{6b^2}{1-c} \]

\[ (1-c) = \frac{b^2}{1-c} \]

\[ (1-c)^2 = b^2 \]

\[ 1 - c = \pm b \]

\[ c = 1 \pm b \]

**Blunders (-3)**
B1 Indices
B2 Only 1 value of c given
B3 Factors
1 (c)

\((x - c + 1)\) is a factor of \(f(x)\)  \(\implies (c - 1)\) is a root
\(\implies f(c - 1) = 0\)

\(f(x) = x^2 - 5x + 5cx - 6b^2\)
\(f(c - 1) = (c - 1)^2 - 5(c - 1) + 5c(c - 1) - 6b^2 = 0\)
\(c^2 - 2c + 1 - 5c + 5 + 5c^2 - 5c = 6b^2\)
\(6c^2 - 12c + 6 = 6b^2\)
\(6(c^2 - 2c + 1) = 6(b^3)\)
\((c - 1)^2 = b^2\)
\(c - 1 = \pm b\)
\(c = 1 \pm b\)

Blunders (-3)
B1 Indices
B2 Expansion of \((c - 1)^2\) once only
B3 Only 1 value of \(c\) given
B4 Factors
QUESTION 2

Part (a) 10 (5, 5) marks  Att (2, 2)
Part (b) 20 (10, 5, 5) marks  Att (3, 2, 2)
Part (c) 20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

2. (a) Solve the simultaneous equations

\[
\begin{align*}
&x - y + 8 = 0 \\
&x^2 + xy + 8 = 0.
\end{align*}
\]

Quadratic Values 5 marks  Att 2

\[
\begin{array}{|c|}
\hline
2 (a) & x = y - 8. \therefore (y - 8)^2 + y(y - 8) + 8 = 0. \\
&y^2 - 16y + 64 + y^2 - 8y + 8 = 0 \\
&2y^2 - 24y + 72 = 0 \Rightarrow y^2 - 12y + 36 = 0. \\
&(y - 6)^2 = 0 \Rightarrow y = 6. \\
&\therefore \text{Solution is } (-2, 6). \\
\hline
\end{array}
\]

Blunders (-3)
B1 Indices  
B2 Factors once only  
B3 Deduction value from factor  
B4 Not getting 2nd value (having got 1st)  
B5 Roots formula once only

Slips (-1)
S1 Numerical

Attempts
A1 Not quadratic

Worthless
W1 Trial and error
(b) (i) The graphs of three quadratic functions, \( f, g \) and \( h \), are shown.

In each case, state the nature of the roots of the function.

(ii) The equation \( kx^2 + (1 - k)x + k = 0 \) has equal real roots.
    Find the possible values of \( k \).

(b) (i) 10 marks Att 3

\[
\begin{array}{|c|}
\hline
\text{(b) (i)} & \text{10 marks} & \text{Att 3} \\
\hline
2 \text{ (b) (i)} & f(x) \text{ has no real roots; (it has two complex roots).} & \\
& g(x) \text{ has two equal real roots. [or: } g(x) \text{ has one real root]} & \\
& h(x) \text{ has two distinct real roots.} & \\
\hline
\end{array}
\]

**Blunders (-3)**

B1 Does not state nature of roots, or states incorrect nature of roots.
B2 Does not state number of roots (once only).

Note: One blunder only in each function

(b) (ii) Quadratic 5 marks Att 2

Values of \( k \)

Equal roots \( \Rightarrow b^2 - 4ac = 0 \).
\[
\therefore (1 - k)^2 - 4k^2 = 0.
\]
\[
1 - 2k + k^2 - 4k^2 = 0 \Rightarrow 3k^2 + 2k - 1 = 0.
\]
\[
(k + 1)(3k - 1) = 0 \Rightarrow k = -1, \ k = \frac{1}{3}.
\]

**Blunders (-3)**

B1 Indices
B2 Real equal roots condition
B3 Factors once only
B4 Roots formula once only
B5 Deduction of value from factor or no value from factor
Part (c)  20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

(c) (i) One of the roots of $px^2 + qx + r = 0$ is $n$ times the other root.
Express $r$ in terms of $p, q$ and $n$.

(ii) One of the roots of $x^2 + qx + r = 0$ is five times the other.
If $q$ and $r$ are positive integers, determine the set of possible values of $q$.

(c) (i) Root 5 marks Att 2
Express $r$ 5 marks Att 2

**2 (c) (i)**

Roots are $\alpha$ and $n\alpha$.
\[ \alpha + n\alpha = -\frac{q}{p} \quad \text{and} \quad \alpha(n\alpha) = \frac{r}{p}. \]
\[ \alpha(1 + n) = -\frac{q}{p} \Rightarrow \alpha = \frac{-q}{p(1 + n)}. \]
But $\alpha^2 = \frac{r}{pn}$
\[ \Rightarrow \quad \frac{q^2}{p^2(1 + n)^2} = \frac{r}{pn}. \]
\[ \therefore \quad r = \frac{nq^2}{p(n + 1)^2}. \]

(c) (ii) $r$ in terms of $q$ 5 marks Att 2
Values of $q$ 5 marks Att 2

**2 (c) (ii)**

\[ r = \frac{nq^2}{p(n + 1)}, \text{ by part (i), where } n = 5 \text{ and } p = 1. \]
\[ \therefore \quad r = \frac{5q^2}{36}. \]
For $r$ to be a positive integer, $q^2$ must be divisible by 36, so $q$ is divisible by 6.
\[ \therefore \quad q = \{6, 12, 18, 24, \ldots \}. \]

**OR**

2 (c) (ii) Equation : $x^2 - (-q)x + (r) = 0$

Roots : $\alpha, 5\alpha$
\[ x^2 - (\alpha + 5\alpha)x + (5\alpha^2) = 0 \]

Equating Coefficients: (i) : $6\alpha = -q \Rightarrow \alpha = -\frac{q}{6}$

(ii) $5\alpha^2 = r$
\[ 5\left(-\frac{q}{6}\right)^2 = r \]
\[ r = \frac{5q^2}{36}. \]

For $r$ to be a positive integer, $q^2$ must be divisible by 36, so $q$ is divisible by 6.
\[ \therefore \quad q = \{6, 12, 18, 24, \ldots \}. \]
**Blunders** (-3)

B1  Indices
B2  Statement quadratic equation once only
B3  Incorrect sum roots
B4  Incorrect product roots
B5  One value of \( q \) only or two values \( q \)

**Slips** (-1)

S1  Numerical
## QUESTION 3

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

### Part (a) 10 (5, 5) marks Att (2, 2)

3 (a)  
\[ z_1 = a + bi \text{ and } z_2 = c + di, \text{ where } i^2 = -1. \]

Show that \[ z_1 + z_2 = \overline{z_1} + \overline{z_2}, \] where \( \overline{z} \) is the complex conjugate of \( z \).

| \( \overline{z_1} + \overline{z_2} \) | 5 marks | Att 2 |
| \( z_1 + z_2 \) | 5 marks | Att 2 |

3 (a) \[ \overline{z_1} = a - bi, \overline{z_2} = c - di \Rightarrow \overline{z_1} + \overline{z_2} = (a + c) - (b + d)i. \]

\[ z_1 + z_2 = (a + c) + (b + d)i = (a + c) - (b + d)i = \overline{z_1} + \overline{z_2}. \]

**Blunders (-3)**

B1 \( i \)

B2 Conjugate
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(b) Let \( A = \frac{1}{2} \begin{pmatrix} 1 & 1 - \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \)

(i) Express \( A^3 \) in the form \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \), where \( a, b \in \mathbb{Z} \).

(ii) Hence, or otherwise, find \( A^{17} \).


<table>
<thead>
<tr>
<th></th>
<th>(b) (i) ( A^2 )</th>
<th>5 marks</th>
<th>Att 2</th>
<th></th>
<th>(b) (i) ( A^3 )</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( A^2 = \frac{1}{4} \begin{pmatrix} 1 &amp; 1 - \sqrt{3} \ \sqrt{3} &amp; 1 \end{pmatrix} \begin{pmatrix} 1 &amp; 1 - \sqrt{3} \ \sqrt{3} &amp; 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 &amp; -2 \sqrt{3} \ 2 \sqrt{3} &amp; -2 \end{pmatrix} )</td>
<td></td>
<td></td>
<td></td>
<td>( A^3 = \frac{1}{8} \begin{pmatrix} 1 &amp; 1 - \sqrt{3} \ \sqrt{3} &amp; 1 \end{pmatrix} \begin{pmatrix} -2 &amp; -2 \sqrt{3} \ 2 \sqrt{3} &amp; -2 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -8 &amp; 0 \ 0 &amp; -8 \end{pmatrix} = \begin{pmatrix} -1 &amp; 0 \ 0 &amp; -1 \end{pmatrix} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) (ii) Values in \( A^{17} \) 5 marks Att 2

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>(b) (ii) ( A^{17} ) calculated</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>
|   | \( A^{17} = \left( A^3 \right)^5 A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 - \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 \sqrt{3} \\ 2 \sqrt{3} & -2 \end{pmatrix} \) |   |   |   | \( = \frac{1}{4} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & -2 \sqrt{3} \\ 2 \sqrt{3} & -2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \sqrt{3} \\ -2 \sqrt{3} & 2 \end{pmatrix} \) |   |   | \( = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \).

Blunders (-3)

B1 Indices

Slips (-1)

S1 Numerical

S2 Each incorrect element

Note: Can only get Att 2 in (ii) if \( A^3 \) not a diagonal matrix (in second 5 marks).
(c) (i) Use De Moivre’s theorem to prove that $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$.

(ii) Hence, find $\int \sin^3 \theta \, d\theta$.

(c) (i) $\sin 3\theta$ 5 marks  
Value 5 marks

3 (c) (i)

\[
(\cos \theta + i\sin \theta)^3 = \cos 3\theta + i\sin 3\theta.
\]

\[
(\cos \theta + i\sin \theta)^3 = \cos^3 \theta + 3\cos^2 \theta (i\sin \theta) + 3\cos \theta (i\sin \theta)^2 + (i\sin \theta)^3.
\]

\[
= \cos^3 \theta - 3\cos \theta \sin^2 \theta + 3\cos^2 \theta \sin \theta - i\sin^3 \theta.
\]

$\therefore$ $\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta = 3\sin \theta (1 - \sin^2 \theta) - \sin^3 \theta$

\[
= 3\sin \theta - 4\sin^3 \theta.
\]

(c) (ii) $\int \sin^3 \theta \, d\theta$ 5 marks  
Finish 5 marks

3 (c) (ii)

\[
\sin 3\theta = 3\sin \theta - 4\sin^3 \theta \Rightarrow \sin^3 \theta = \frac{1}{4} [3\sin \theta - 3\sin 3\theta]
\]

$\therefore \int \sin^3 \theta \, d\theta = \frac{1}{4} \int (3\sin \theta - 3\sin 3\theta) \, d\theta = \frac{1}{4} \left[ -3\cos \theta + \frac{1}{3} \cos 3\theta \right] + C.$

Note: Not “hence” $\Rightarrow$ zero marks for integration.

Blunders (-3)
B1 Statement De Moivre once only
B2 Binomial expansion once only
B3 $i$
B4 Indices
B5 Trig formula
B6 Not like to like when equating coefficients
B7 Integration
B8 $C$ omitted
# QUESTION 4

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

### Part (a) 10 (5, 5) marks Att (2, 2)

4. (a) Three consecutive terms of an arithmetic series are \(4x + 11\), \(2x + 11\), and \(3x + 17\). Find the value of \(x\).

**Definition of A.P.** 5 marks Att 2

**Value** \(x\) 5 marks Att 2

\[\begin{align*}
(2x + 11) - (4x + 11) &= (3x + 17) - (2x + 11) \\
-2x &= x + 6 \\
&
\Rightarrow x = -2.
\end{align*}\]

(And the three terms are 3, 7 and 11.)

### Blunders (-3)

B1  AP statement

### Slips (-1)

S1  Numerical

### Worthless

W1  Geometric sequence

W2  Puts in values for \(x\)

### Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(b) (i) Show that \(\frac{2}{r^2 - 1} = \frac{1}{r - 1} - \frac{1}{r + 1}\), where \(r \neq \pm 1\).

(ii) Hence, find \(\sum_{r=2}^{n} \frac{2}{r^2 - 1}\).

(iii) Hence, evaluate \(\sum_{r=2}^{\infty} \frac{2}{r^2 - 1}\).

\[\begin{align*}
\frac{1}{r - 1} - \frac{1}{r + 1} &= \frac{r + 1 - r + 1}{(r - 1)(r + 1)} = \frac{2}{r^2 - 1}.
\end{align*}\]

**OR**

\[\begin{align*}
\frac{1}{r - 1} - \frac{1}{r + 1} &= \frac{r + 1 - r + 1}{(r - 1)(r + 1)} = \frac{2}{r^2 - 1}.
\end{align*}\]

*Page 19 of 82*
4 (b) (i)

Let \[ \frac{2}{r^2 - 1} = \frac{a}{r - 1} - \frac{b}{r + 1} \]

\[ 2 = q(r + 1) - b(r - 1) \]

\[ (0)r + (2) = (a - b)r + (a + b) \]

Equating Coefficients:

(i) \[ a - b = 0 \]

(ii) \[ a + b = 2 \]

(i) \[ a - b = 0 \]

(ii) \[ a + b = 2 \]

\[ 2a = 2 \]

\[ a = 1 \]

So, \[ a - b = 0 \Rightarrow a = b \Rightarrow a = b = 1 \]

\[ \frac{2}{r^2 - 1} = \frac{1}{r - 1} - \frac{1}{r + 1} \]

(b)(ii) Set up cancellation 5 marks

Finish 5 marks

4 (b) (ii)

\[ \sum_{r=2}^{n} \frac{2}{r^2 - 1} = \sum_{r=2}^{n} \left( \frac{1}{r - 1} - \frac{1}{r + 1} \right) \]

\[ = \sum_{r=2}^{n} \left( \frac{1}{r - 1} \right) - \sum_{r=2}^{n} \left( \frac{1}{r + 1} \right) \]

\[ = \sum_{r=2}^{n} \frac{1}{r} - \sum_{r=2}^{n} \frac{1}{r} \]

\[ = \left( 1 + \frac{1}{2} + \sum_{r=3}^{n-1} \right) - \left( \sum_{r=2}^{n-1} \frac{1}{r} + \frac{1}{n} + \frac{1}{n+1} \right) \]

\[ = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \]

OR
**4 (b) (ii)**

Terms $U_2$ to $U_n$  
Sum to $n$ terms  

<table>
<thead>
<tr>
<th>Term</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_n$</td>
<td>$\frac{1}{n^2 - 1}$</td>
</tr>
<tr>
<td>$U_{n-1}$</td>
<td>$\frac{1}{n-2} - \frac{1}{n}$</td>
</tr>
<tr>
<td>$U_{n-2}$</td>
<td>$\frac{1}{n-3} - \frac{1}{n-1}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$U_4$</td>
<td>$\frac{1}{3} - \frac{1}{5}$</td>
</tr>
<tr>
<td>$U_3$</td>
<td>$\frac{1}{2} - \frac{1}{4}$</td>
</tr>
<tr>
<td>$U_2$</td>
<td>$1 - \frac{1}{3}$</td>
</tr>
</tbody>
</table>

$$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

**4 (b) (iii)**

Sum to infinity  

$$\sum_{r=2}^{n-1} \frac{2}{r^2 - 1} = \text{Limit}_{\substack{n \to \infty}} \left( \frac{3}{2} \cdot \frac{1}{n} - \frac{1}{n+1} \right) = \frac{3}{2}.$$  

**Blunders (-3)**  
B1 Indices  
B2 Cancellation must be shown or implied  
B3 Not like to like when equating coefficients  
B4 Term omitted  
B5 Gets $S_r$

**Slips (-1)**  
S1 Numerical

Note: Must show three terms at start and two terms at finish or *vice versa*. 
Part (c)  

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

(c) A finite geometric sequence has first term $a$ and common ratio $r$. The sequence has $2m + 1$ terms, where $m \in \mathbb{N}$.

(i) Write down the last term, in terms of $a$, $r$, and $m$.

(ii) Write down the middle term, in terms of $a$, $r$, and $m$.

(iii) Show that the product of all of the terms of the sequence is equal to the middle term raised to the power of the number of terms.

<table>
<thead>
<tr>
<th>Part (c) (i)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c) (i)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 (c) (i)</td>
<td>Last term = $ar^{2m}$.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Part (c) (ii)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c) (ii)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 (c) (ii)</td>
<td>Middle term = $ar^m$.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(c) (iii) Product</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c) (iii) Show</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 4 (c) (iii)       | Product of terms = $a \times ar \times ar^2 \times \ldots \times ar^{2m}$
|                   | = $a^{2m+1} \times r^{0+1+2+\ldots+2m}$.
|                   | $[0 + 1 + 2 + \ldots + 2m$ is an A.P. with 2$m+1$ terms $]$.
|                   | = $a^{2m+1} \left(r \frac{(2m+1)}{2} \frac{(2m)}{2} \right) = a^{2m+1} \cdot m \cdot (2m+1)$.
|                   | = $(ar^m)^{2m+1}$. |       |

Blunders (-3)
- B1 Indices
- B2 $U_n \neq AR^{n-1}$
- B3 Formula AP
- B4 Incorrect substitution into formula once only
- B5 Middle term

Slips (-1)
- S1 Numerical
**QUESTION 5**

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 10) marks</td>
<td>Att (2, 2, 3)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

**Part (a) 10 (5, 5) marks  Att (2, 2)**

5 (a) Solve for: \( x - 2 = \sqrt{3x - 2} \).

### Quadratic Solution 5 marks  Att 2

<table>
<thead>
<tr>
<th>Quadratic</th>
<th>Solution 5 marks  Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x - 2 = \sqrt{3x - 2} ) ( \Rightarrow (x - 2)^2 = 3x - 2 ).</td>
<td></td>
</tr>
<tr>
<td>( x^2 - 4x + 4 = 3x - 2 ) ( \Rightarrow x^2 - 7x + 6 = 0 ).</td>
<td></td>
</tr>
<tr>
<td>( (x - 6)(x - 1) = 0 ) ( \Rightarrow x = 6 ) and ( x = 1 ).</td>
<td></td>
</tr>
</tbody>
</table>

Test: \( x = 1 \)

LHS: \( (x - 2) = (1 - 2) = -1 \)

RHS: \( \sqrt{3x - 2} = \sqrt{1} = 1 \)

\( x \neq 1 \)

\( x = 6 \)

LHS: \( x - 2 = 6 - 2 = 4 \)

RHS: \( \sqrt{3x - 2} = \sqrt{16} = 4 \)

Solution: \( x = 6 \)

**Blunders (-3)**

B1 Indices

B2 Expansion \((x - 2)^2\) once only

B3 Factors once only

B4 Roots formula once only

B5 Deduction value from factor

B6 Excess value

**Slips (-1)**

S1 Numerical

**Attempts**

A1 \( x = 6 \) and no other work merits Att 2

A2 \( x = 6 \) by trial and error merits Att 2
Part (b) 20 (5, 5, 10) marks Att (2, 2, 3)

(b) Prove by induction that, for all positive integers \( n \), 5 is a factor of \( n^5 - n \).

**P(1)** 5 marks Att 2

**P(k)** 5 marks Att 2

**P(k+1)** 10 marks Att 3

Let \( P(n) \) be the proposition that 5 is a factor of \( n^5 - n \).

Test \( P(1) \): \( 1^5 - 1 = 0 \), which is divisible by 5.

Assume \( P(k) \): \( k^5 - k \) is divisible by 5.

Try to deduce \( P(k+1) \): that \( (k+1)^5 - (k+1) \) is divisible by 5.

\[
(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1
\]

\[
= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)
\]

\[
\text{div. by 5 from } P(k) \text{ has 5 as factor}
\]

so sum is divisible by 5, given \( P(k) \).

We have \( P(1) \) and \( \{P(k) \Rightarrow P(k+1)\} \). Hence, \( P(n) \) for all positive integers \( n \).

**OR**

To prove: \( (n^5 - n) \) is divisible by 5.

\( n = 1: 1^5 - 1 = 0 \), which is divisible by 5

\( \Rightarrow \) true for \( n = 1 \).

Assume true for \( n = k \): \( k^5 - k \) is divisible by 5.

To prove: \( (k+1)^5 - (k+1) \) is divisible by 5.

Let \( f(k) = k^5 - k \). Given the assumption that \( f(k) \) is divisible by 5, then \( f(k+1) \) will be divisible by 5 if and only if \( [f(k+1) - f(k)] \) is divisible by 5.

Now, \( f(k+1) - f(k) = [(k+1)^5 - (k+1)] - [k^5 - k] \)

\[
= [(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - k - 1] - k^5 + k
\]

\[
= 5k^4 + 10k^3 + 10k^2 + 5k
\]

\[
= 5(k^4 + 2k^3 + 2k^2 + k), \text{ which is divisible by 5.}
\]

So, the statement is true for \( n = k + 1 \) whenever it is true for \( n = k \).

Since it is true for \( n = 1 \), then, by induction, it is true for all positive integers.

**Blunders (-3)**

B1 Binomial expansion once only
B2 Indices
B3 Expansion of \( (k+1)^5 \) once only

Note: Must prove \( P(1) \) step (not sufficient to state \( P(n) \) true for \( n = 1 \)).
(c) Solve the simultaneous equations
\[
\log_3 x + \log_3 y = 2 \\
\log_3 (2y - 3) - 2\log_3 x = 1.
\]

One var. in terms of the other 5 marks  Att 2
Change of base 5 marks  Att 2
Quadratic 5 marks  Att 2
Solution 5 marks  Att 2

5 (c)
\[
\begin{align*}
\log_3 x + \log_3 y &= 2 \\
\log_3 (xy) &= 2 \\
x &= 9 \\
y &= \frac{9}{x}
\end{align*}
\]

\[
\begin{align*}
\log_3 (2y - 3) - 2\log_3 x &= 1 \\
\log_3 \left(\frac{2y - 3}{x^2}\right) &= 1 \\
\frac{2y - 3}{x^2} &= 3
\end{align*}
\]

\[
\begin{align*}
(2y - 3) = 3 \\
2y^2 - 3y - 27 &= 0 \\
(2y - 9)(y + 3) &= 0
\end{align*}
\]

\[
y > 0 \Rightarrow y \neq -3 \text{, so } y = \frac{9}{2} \text{, giving } x = 2.
\]

Blunders (-3)
B1 Logs
B2 Indices
B3 Formula change of base
B4 Factors
B5 Roots formula
B6 Deduction root from factor or no deduction
B7 Excess value

Worthless
W1 Drops “logs”

Note Must have a quadratic equation for last 5 marks
QUESTION 6

Part (a) 10 marks Att 3

6 (a) Differentiate $\sin(3x^2 - x)$ with respect to $x$.

$$f(x) = \sin(3x^2 - x) \Rightarrow f'(x) = \cos(3x^2 - x)(6x - 1)$$

Blunders (-3)
B1 Differentiation

Attempts
A1 Error in differentiation formula

Part (b) 15 (5, 5, 5) marks Att (2, 2, 2)

(b)(i) Differentiate $\sqrt{x}$ with respect to $x$, from first principles.

(ii) An object moves in a straight line such that its distance from a fixed point is given by $s = \sqrt{t^2 + 1}$, where $s$ is in metres and $t$ is in seconds. Find the speed of the object when $t = 5$ seconds.

(b)(i) $f(x + h) - f(x)$

Multiplication 5 marks Att 2
Finish 5 marks Att 2

6 (b) (i)

$$f(x) = \sqrt{x} \Rightarrow f(x) = \sqrt{x + h}$$

$$f(x + h) - f(x) = \sqrt{x + h} - \sqrt{x}$$

$$= \frac{(\sqrt{x + h} - \sqrt{x}) (\sqrt{x + h} + \sqrt{x})}{\sqrt{x + h} + \sqrt{x}}$$

$$= \frac{x + h - x}{\sqrt{x + h} + \sqrt{x}}$$

$$= \frac{h}{\sqrt{x + h} + \sqrt{x}}$$

$$\therefore \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x + h + \sqrt{x}}} = \frac{1}{2\sqrt{x}}$$

OR
6 (b) (i)

\[ y = \sqrt{x} \]
\[ y + \Delta y = \sqrt{x + \Delta x} \]
\[ \Delta y = \sqrt{x + \Delta x} - \sqrt{x} \]
\[ \frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \]
\[ = \frac{(\sqrt{x + \Delta x})^2 - (\sqrt{x})^2}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \]
\[ = \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \]
\[ = \frac{1}{\Delta x} \cdot \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \]
\[ \text{Limit as } \Delta x \to 0 \]
\[ \frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \]
\[ = \frac{1}{\sqrt{x + \sqrt{x}}} = \frac{1}{2\sqrt{x}} \]

Blunders (-3)
B1 Differentiation
B2 Indices
B3 No limits shown or implied or no indication \( h \to 0 \)
B4 \( h \to \infty \)
B5 Conjugate
B6 No left hand side

Worthless
W1 Not 1st principles

(b) (ii) 5 marks Att 2

\[ s = (t^2 + 1)^{1.5} \Rightarrow \frac{ds}{dt} = 1.5 \cdot \frac{2t}{\sqrt{t^2 + 1}} \]
\[ \Rightarrow 2. t = \frac{t}{\sqrt{t^2 + 1}} \]
\[ \therefore \text{At } t = 5, \frac{ds}{dt} = \frac{5}{\sqrt{26}} \text{ metres per second.} \]

Blunders (-3)
B1 Differentiation
B2 Indices
B3 No substitution \( t = 5 \)

Slips (-1)
S1 Incorrect units or omitted units

Attempts
A1 Error in differentiation formula
The equation of a curve is \[ y = \frac{2}{x-3}. \]

(i) Write down the equations of the asymptotes and hence sketch the curve.

(ii) Prove that no two tangents to the curve are perpendicular to each other.

(i) Asymptotes

Equations of asymptotes are \( x = 3 \) and \( y = 0 \).

\[ y = \frac{2}{x-3} = 2(x-3)^{-1} \Rightarrow \frac{dy}{dx} = -2(x-3)^{-2} = \frac{-2}{(x-3)^2}. \]

\[
\therefore \text{ Slope of tangent at } (x, y) \text{ is } m = \frac{-2}{(x-3)^2}.
\]

But \( m \) will be negative for all values of \( x \Rightarrow m_1m_2 \neq -1 \)

\[
\therefore \text{ No two tangents are perpendicular to each other.}
\]

OR
6 (c) (ii)

\[ y = 2(x - 3)^3 \]

\[ m = \frac{dy}{dx} = -\frac{2}{(x-3)^2} \]

Let tangents at \( x = a \) and \( x = b \) be perpendicular

At \( x = a \) : \( m_1 = -\frac{2}{(a-3)^2} \)

At \( x = b \) : \( m_2 = -\frac{2}{(b-3)^2} \)

\[ (m_1)(m_2) = \frac{-2}{(a-3)^2} \cdot \frac{-2}{(b-3)^2} = \frac{4}{(a-3)^2(b-3)^2} \neq -1, \text{ (since LHS is positive).} \]

\[ \Rightarrow \text{Tangents cannot be perpendicular.} \]

**Blunders (-3)**
- B1 Indices
- B2 Asymptote
- B3 Differentiation
- B4 Slope \( \neq \frac{dy}{dx} \)
- B5 \( m_1m_2 \neq -1 \)
- B6 Incorrect deduction or no deduction

**Slips**
- S1 Curve not approaching asymptotes.

**Attempts**
- A1 Error in differentiation formula

**Worthless**
- W1 Integration
QUESTION 7

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

7 (a) The equation of a curve is \( x^2 - y^2 = 25 \). Find \( \frac{dy}{dx} \) in terms of \( x \) and \( y \).

Differentiate 5 marks Att 2

Isolate \( \frac{dy}{dx} \) 5 marks Att 2

\[
7 (a) \quad x^2 - y^2 = 25 \implies 2x - 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{x}{y}.
\]

OR

\[
7 (a) \quad x^2 - y^2 = 25 \quad \text{OR} \quad y = \sqrt{x^2 - 25}
\]

\[
y = \left(x^2 - 25\right)^{\frac{1}{2}}
\]

\[
\frac{dy}{dx} = \frac{1}{2} \left(x^2 - 25\right)^{-\frac{1}{2}} \cdot 2x
\]

\[
= \frac{x}{\sqrt{x^2 - 25}}
\]

\[
= \frac{x}{y}
\]

\[
\frac{dy}{dx} = \frac{x}{y}
\]

Blunders (-3)
B1 Differentiation
B2 Indices

Attempts
A1 Error in differentiation formula
A2 \( \frac{dy}{dx} = 2x - 2y \frac{dy}{dx} \) and uses two \( \frac{dy}{dx} \) terms in first 5 marks.

Worthless
W1 No differentiation
W2 Integration
Part (b)  

A curve is defined by the parametric equations 
\[ x = \frac{3t}{t^2 - 2} \quad \text{and} \quad y = \frac{6}{t^2 - 2}, \] where \( t \neq \pm \sqrt{2} \).

(i) Find \( \frac{dy}{dx} \) in terms of \( t \).

(ii) Find the equation of the tangent to the curve at the point given by \( t = 2 \).

\[
\begin{align*}
\text{(b) (i)} & \quad \frac{dx}{dt}, \frac{dy}{dt} \quad 5 \text{ marks} \\
\frac{dy}{dx} & \quad 5 \text{ marks}
\end{align*}
\]

\[
\begin{align*}
7 \ (b) \ (i) & \quad x = \frac{3t}{t^2 - 2} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{3(t^2 - 2) - 3t \cdot 2t}{(t^2 - 2)^2} = \frac{-3t^2 - 6}{(t^2 - 2)^2}. \\
& \quad y = \frac{6}{t^2 - 2} = 6(t^2 - 2)^{-1} \quad \Rightarrow \quad \frac{dy}{dt} = -6(t^2 - 2)^{-2} \cdot 2t = \frac{-12t}{(t^2 - 2)^2}. \\
& \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-12t}{(t^2 - 2)^2} \cdot \frac{3t^2 + 6}{3t^2 - 6} = \frac{4t}{t^2 + 2}.
\end{align*}
\]

\[
\begin{align*}
(b) (ii) & \quad \text{Slope, point} \quad 5 \text{ marks} \\
& \quad \text{Equation} \quad 5 \text{ marks}
\end{align*}
\]

\[
\begin{align*}
7 \ (b) \ (ii) & \quad t = 2 \quad \Rightarrow \quad x = \frac{6}{2} = 3 \quad \text{and} \quad t = 2 \quad \Rightarrow \quad y = \frac{6}{2} = 3. \quad :. \quad \text{Point is} \quad (3, 3).
\end{align*}
\]

Slope of tangent at \( t = 2 \) is \( \frac{8}{6} = \frac{4}{3} \).

:. Equation of tangent: \( y - 3 = \frac{4}{3}(x - 3) \quad \Rightarrow \quad 4x - 3y - 3 = 0. \)

**Blunders (-3)**
B1 Differentiation
B2 Indices
B3 Error in getting \( \frac{dy}{dx} \)
B4 Equation of tangent
B5 Error in slope formula.

**Slips (-1)**
S1 Numerical

**Attempts**
A1 Error in differentiation formula
The function \( f(x) = x^3 - 3x^2 + 3x - 4 \) has only one real root.

(i) Show that the root lies between 2 and 3.

Anne and Barry are each using the Newton-Raphson method to approximate the root. Anne is starting with 2 as a first approximation and Barry is starting with 3.

(ii) Show that Anne’s starting approximation is closer to the root than Barry’s. (That is, show that the root is less than 2.5.)

(iii) Show, however, that Barry’s next approximation is closer to the root than Anne’s.

\[
f(x) = x^3 - 3x^2 + 3x - 4.
\]

\[
f(2) = 8 - 12 + 6 - 4 = -2 < 0.
\]

\[
f(3) = 27 - 27 + 9 - 4 = 5 > 0.
\]

\[
\therefore \text{ root lies between 2 and 3.}
\]

\[
f(2.5) = (2.5)^3 - 3(2.5)^2 + 3(2.5) - 4
\]

\[
= 15.625 - 18.75 + 7.5 - 4
\]

\[
= 0.375
\]

\[
f(2) < 0 \quad \text{and} \quad f(2.5) > 0 \implies \text{ root is between 2 and 2.5.}
\]

So, root is closer to 2 than to 3.

Ann:
\[
f(2) = -2 \quad \text{and} \quad f'(2) = 3.
\]

\[
x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-2}{3} = 2 \frac{2}{3} = 2.666...
\]

Barry:
\[
f(3) = 5 \quad \text{and} \quad f'(3) = 12.
\]

\[
x_1 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{5}{12} = 3 - 0.416... = 2.583...
\]

Both of these are above the root, so the lower one is closer (i.e. Barry’s).

**Blunders** (-3)

B1 Indices
B2 Incorrect deduction from \( f(2) \) and \( f(3) \) or no deduction
B3 No \( f(2.5) \)
B4 Newton–Raphson formula
B5 Differentiation
B6 Incorrect deduction or no deduction from work in (iii)
# QUESTION 8

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>8 (a)</strong></td>
<td>Find $\int \left( 6x + 3 + \frac{1}{x^2} \right) dx$.</td>
<td></td>
</tr>
</tbody>
</table>

\[
\int \left( 6x + 3 + \frac{1}{x^2} \right) dx = 3x^2 + 3x - \frac{1}{x} + C.
\]

**Blunders** (-3)
B1 Integration
B2 Indices
B3 No c

**Attempts**
A1 Only c correct

**Worthless**
W1 Differentiation for integration

<table>
<thead>
<tr>
<th>Part (b)</th>
<th>20 (5, 5, 5, 5) marks</th>
<th>Att (2, 2, 2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(b)</strong> Evaluate</td>
<td>(i) $\int_{\pi/4}^{\pi} \sin 3x \sin x , dx$</td>
<td>(ii) $\int_{\ln 3}^{\ln 8} e^{x} \sqrt{1 + e^{x}} , dx$.</td>
</tr>
</tbody>
</table>

### Integration

<table>
<thead>
<tr>
<th>Value</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>8 (b) (i)</strong></td>
<td>$\int_{\pi/4}^{\pi/4} \sin 3x \sin x , dx = \frac{1}{2} \int_{\pi/4}^{\pi} \left( \cos 2x - \cos 4x \right) dx$</td>
<td>$\frac{\pi}{4}$</td>
</tr>
</tbody>
</table>

\[
\int_{\pi/4}^{\pi} \left( \cos 2x - \cos 4x \right) dx = \frac{1}{2} \int_{\pi/4}^{\pi} \left( \frac{1}{2} \sin 2x - \frac{1}{4} \sin 4x \right) dx
\]

\[
= \frac{1}{2} \left[ \left( \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{4} \sin \pi \right) - \left( \frac{1}{2} \sin \left( -\frac{\pi}{2} \right) - \frac{1}{4} \sin (-\pi) \right) \right]
\]

\[
= \frac{1}{2} \left[ (1 - 0) - (-1 - 0) \right] = \frac{1}{2}.
\]

Page 33 of 82
Integration Value

8 (b) (ii) Let \( u = 1 + e^x \). \( \therefore du = e^x \, dx \).
\[
\int e^x \sqrt{1 + e^x} \, dx = \int_{\ln3}^{1 + \ln8} u^{\frac{1}{2}} \, du , \text{ but } e^{\ln8} = 8 \text{ and } e^{\ln3} = 3.
\]
\[
= \int_{4}^{9} u^{\frac{1}{2}} \, du = \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_{4}^{9} = \frac{2}{3} [27 - 8] = \frac{38}{3}.
\]

OR

Using \( x \) limits:
\[
\int_{\ln3}^{\ln8} e^x \sqrt{1 + e^x} \, dx = \frac{2}{3} u^{\frac{3}{2}} \bigg|_{\ln3}^{\ln8}
\]
\[
= \frac{2}{3} \left( 1 + e^{\ln8} \right)^{\frac{3}{2}} - \frac{2}{3} \left( 1 + e^{\ln3} \right)^{\frac{3}{2}}
\]
\[
= \frac{2}{3} \left[ (9)^{\frac{3}{2}} - (4)^{\frac{3}{2}} \right] = \frac{2}{3} [27 - 8] = \frac{38}{3}.
\]

OR

\[
\int_{\ln3}^{\ln8} e^x \sqrt{1 + e^x} \, dx \quad \text{Let } u = e^x
\]
\[
= \int (1 + u) \frac{1}{2} \, du
du = e^x \, dx
\]
\[
= \frac{2}{3} \left( 1 + u \right)^{\frac{3}{2}} \bigg|_{\ln3}^{\ln8}
\]
\[
= \frac{2}{3} \left( 1 + e^{\ln8} \right)^{\frac{3}{2}} - \frac{2}{3} \left( 1 + e^{\ln3} \right)^{\frac{3}{2}}
\]
\[
= \frac{2}{3} \left[ (1 + 8)^{\frac{3}{2}} - (1 + 3)^{\frac{3}{2}} \right] = \frac{2}{3} [27 - 8] = \frac{38}{3}.
\]

* Incorrect substitution and unable to finish yields attempt at most
Blunders (-3)
B1 Trig formula
B2 Integration
B3 Differentiation
B4 Limits
B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits
B8 Indices
B9 Logs
B10 $e^{\ln a} \neq a$

Slips (-1)
S1 Numerical
S2 Trig value
S3 Answer not tidied up

Worthless
W1 Differentiation instead of integration except where other work merits attempts
Part (c) 20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

(c) Use integration methods to establish the standard formula for the volume of a cone.

<table>
<thead>
<tr>
<th>Description</th>
<th>Marks</th>
<th>Attitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diagram + slope</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Correct subst. into volume formula</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Integration</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Volume</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

**8 (c)**

\[
y = mx \quad \Rightarrow \quad y = \frac{r}{h} x.
\]

Volume of cone = \( \pi \int_0^h y^2 \, dx \), where \( y = \frac{r}{h} x \).

\[
V = \pi \int_0^h \left[ \frac{r^2}{h^2} x^2 \right] \, dx = \frac{1}{3} \pi \frac{r^2}{h^2} h^3
\]

\[
V = \frac{1}{3} \pi r^2 h.
\]

**Blunders (−3)**

B1 Integration
B2 Slope of line
B3 Equation of line
B4 Volume formula provided it is quadratic
B5 Limits
B6 No Limits
B7 Incorrect order in applying limits
B8 Indices

**Slips (−1)**

S1 Numerical

**Attempts**

A1 Uses \( \tau = \tau \)

**Worthless**

W1 Differentiation instead of integration
GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates’ work as follows:
   • Blunders - mathematical errors/omissions (-3)
   • Slips - numerical errors (-1)
   • Misreadings (provided task is not oversimplified) (-1).

Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   • any correct, relevant step in a part of a question merits at least the attempt mark for that part
   • if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   • a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The same error in the same section of a question is penalised once only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most. 

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.
QUESTION 1

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 20 (10, 10) marks Att (3, 3)
Part (c) 20 (10, 5, 5) marks Att (3, 2, 2)

Part (a) 10(5, 5) marks Att (2, 2)

1 (a) Show that, for all values of \( t \in \mathbb{R} \), the point \( \left( \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right) \) lies on the circle \( x^2 + y^2 = 1 \).

Part (a) Substitution 5 marks Att 2
Finish 5 marks Att 2

\[
x^2 + y^2 = \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^2} = \frac{1 + 2t^2 + t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1.
\]

Blunders (-3)
B1 Incorrect squaring (apply once if same type of error)
B2 Incorrect factors
B3 Incorrect conclusion

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 Some correct substitution for \( x \) or \( y \)
A2 Effort at expressing \( t^2 \) in terms of \( y \)

Part (b) 20 (10, 10) marks Att (3, 3)

(b) (i) Find the equation of the tangent to the circle \( x^2 + y^2 = 10 \) at the point \( (3,1) \).

(ii) Find the values of \( k \in \mathbb{R} \) for which the line \( x - y + k = 0 \) is a tangent to the circle \( (x-3)^2 + (y+4)^2 = 50 \).

Part (b) (i) 10 marks Att 3

1 (b) (i) Equation of tangent: \( xx_1 + yy_1 = r^2 \) \( \Rightarrow \) \( 3x + y = 10 \).

or

Centre of circle \((0,0)\) \( \Rightarrow \) Slope diameter = \( \frac{1}{3} \) \( \Rightarrow \) Slope Tangent = -3

Equation of tangent: \( y-1 = -3(x-3) \)
Blunders (-3)
B1 Error in slope formula
B2 Slope of tangent not perpendicular to the diameter
B3 Error in equation of line formula
B4 Error in equation of tangent formula
B5 Incorrect centre of circle

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Equation of tangent formula
A2 Slope of diameter only
A3 Equation of line with some substitution

Part (b) (ii) 10 marks  Att 3

1 (b) (ii) Centre (3, -4) and radius = \sqrt{50} = 5\sqrt{2}.
Since a tangent, perpendicular distance from centre (3, -4) to \( x - y + k = 0 \)
equals radius.
\[
\therefore \left| \frac{3 + 4 + k}{\sqrt{2}} \right| = 5\sqrt{2} \Rightarrow |7 + k| = 10 \Rightarrow 7 + k = \pm 10. \therefore k = 3 \text{ or } k = -17.
\]

OR

Part (b) (ii) 10 marks  Att 3

\[
y = x + k \\
(x - 3)^2 + ((x + k) + 4)^2 = 50 \\
2x^2 + (2 + 2k)x + (8k + 25) = 0 \\
\text{One point of contact} \Rightarrow (2 + 2k)^2 - 4(2 + 8k - 25) = 0 \\
\Rightarrow k^2 + 14k - 51 = 0 \\
\Rightarrow (k - 3)(k + 17) = 0 \\
\Rightarrow k = 3, k = -17
\]

Blunders (-3)
B1 Incorrect centre of circle
B2 Error in perpendicular distance formula
B3 Incorrect radius
B4 One value of \( k \) only
B5 Incorrect squaring
B6 Error in factors

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Centre or radius correct
A2 Some correct substitution into perpendicular formula
A3 Some correct substitution of \( y = x + k \) or equivalent into circle
Part (c) 20 (10, 5, 5) marks  
Att (3, 2, 2)  

(c) Two circles intersect at \( p(2, 0) \) and \( q(-2, 8) \). The distance from the centre of each circle to the common chord \([pq]\) is \(\sqrt{20}\).

Find the equations of the two circles.

Part (c) First equation in \( f \) and \( g \) 10 marks  
Equation in one variable 5 marks  
Finish 5 marks  
Att 3  
Att 2  
Att 2

1 (c)  

Slope \( pq = \frac{8-0}{-2-2} = -2 \) \(\Rightarrow\) slope \( st = \frac{1}{2} \).

\[ \begin{align*} 4 + f &= \frac{1}{2} \Rightarrow g = 2f + 8. 
\end{align*} \]

\[ |st|^2 = 20 \Rightarrow (0+g)^2 + (4+f)^2 = 20 \Rightarrow g^2 + f^2 + 8f = 4 \]
\[ \Rightarrow (2f+8)^2 + f^2 + 8f = 4 \Rightarrow 5f^2 + 40f + 60 = 0. \]

\[ \therefore f^2 + 8f + 12 = 0 \Rightarrow (f+2)(f+6) = 0. \]
\[ f = -2 \Rightarrow g = 4 \quad \text{or} \quad f = -6 \Rightarrow g = -4. \]

\[ \therefore \text{Centres are } (-4, 2) \text{ and } (4, 6), \quad r = \sqrt{40}. \]

Circles are: \((x+4)^2 + (y-2)^2 = 40\) and \((x-4)^2 + (y-6)^2 = 40. \)

or \(x^2 + y^2 + 8x - 4y - 20 = 0\) and \(x^2 + y^2 - 8x - 12y + 12 = 0\)

OR

Part (c) First equation in \( f \) and \( g \) 10 marks  
Equation in one variable 5 marks  
Finish 5 marks  
Att 3  
Att 2  
Att 2

Slope \( pq = \frac{0-8}{2-2} = -2 \)

Equation \( pq: \quad y = -2(x-2) \) or \(2x + y - 4 = 0\)

Perp. distance \((-g, -f)\) to \( pq: \quad \left| -2g - f - 4 \right| = \sqrt{20} \)
\[ \Rightarrow -2g - f - 4 = \pm 10 \Rightarrow 2g + f + 14 = 0 \quad \text{and} \quad 2g + f - 6 = 0 \]

Distance from \((0,4) (= \text{midpoint } pq)\) to \((-g, -f)\) \(\Rightarrow (0+g)^2 + (4+f)^2 = 20 \)

Solving between \(g^2 + (4+f)^2 = 20\) and \(2g + f - 6 = 0\) gives \(g = 4\) and \(f = -2\)

Solving between \(g^2 + (4+f)^2 = 20\) and \(2g + f + 14 = 0\) gives \(g = -4\) and \(f = -6\)

Eq. 1: \(x^2 + y^2 + 8x - 4y + c = 0\)
\((2, 0) \text{ on circle} \Rightarrow c = -20 \Rightarrow x^2 + y^2 + 8x - 4y - 20 = 0\)

Eq2: Same method \(\Rightarrow c = 12 \Rightarrow x^2 + y^2 - 8x - 12y + 12 = 0\)

OR
Part (c) First equation in \( f \) and \( g \) 

**Equation in one variable**

5 marks  

Finish  

5 marks

1 (c)

\[(2,0) \in \text{Circle} \Rightarrow 2^2 + 0 + 2g(2) + 2f(o) + c = 0 \]
\[\Rightarrow 4g + c = -4 \Rightarrow c = -4g - 4 \]
\[(-2,8) \in \text{Circle} \Rightarrow -4g + 16f + c + 68 = 0 \]
\[\Rightarrow -4g + 16f - 4g - 4 + 68 = 0 \Rightarrow g = 2(f + 4) \]
\[s(\text{midpoint}) = (0,4) \]
\[\text{But } \sqrt{g^2 + (4+f)^2} = \sqrt{20} \Rightarrow g^2 + (4+f)^2 = 20 \]
\[\Rightarrow (2(f + 4))^2 + (4 + f)^2 = 20 \Rightarrow 5(f + 4)^2 = 20 \]
\[\Rightarrow (f + 4)^2 = 4 \Rightarrow f + 4 = \pm 2 \Rightarrow f = -6 \text{ and } -2 \]
\[f = -6 \Rightarrow g = -4 \Rightarrow c = 12 \]
\[f = -2 \Rightarrow g = 4 \Rightarrow c = -20 \]

Circles:
\[x^2 + y^2 + 8x - 4y - 20 = 0 \]
\[x^2 + y^2 - 8x - 12y + 12 = 0 \]

**OR**
Part (c) First equation in \(f\) and \(g\)  

10 marks  

Att 3  

Equation in one variable  

5 marks  

Att 2  

Finish  

5 marks  

Att 2  

\[
|pq| = \sqrt{(2 + 2)^2 + (0 - 8)^2} = \sqrt{80} \Rightarrow |ps| = \sqrt{20} \\
|pt|^2 = 20 + 20 = 40 \Rightarrow |pt| = \sqrt{40} \\
\therefore p(2,0) as centre of a circle with radius \sqrt{40} \\
\Rightarrow (x - 2)^2 + y^2 = 40 \\
But (-g, -f) on circle \\
\Rightarrow (-g - 2)^2 + (0 + f)^2 = 40 \\
st is a chord. \\
Slope \ pq = \frac{8 - 2}{-2 - 2} = -2 \Rightarrow slope \ st = \frac{1}{2} \\
\Rightarrow \frac{4 + f}{0 + g} = \frac{1}{2} \Rightarrow g = 2f + 8 \\
\therefore (-2f - 8 - 2)^2 + f^2 = 40 \\
\Rightarrow 5f^2 + 40f + 60 = 0 \Rightarrow f^2 + 8f + 12 = 0 \\
\Rightarrow (f + 6)(f + 2) = 0 \Rightarrow f = -2 and f = -6 \\
f = -6 \Rightarrow g = -4 \Rightarrow c = 12 \\
f = -2 \Rightarrow g = 4 \Rightarrow c = -20 \\
Circles \\
x^2 + y^2 + 8x - 4y - 20 = 0 \\
x^2 + y^2 - 8x - 12y + 12 = 0 \\

Blunders (-3) 
B1 Error in distance formula 
B2 Error in mid point formula 
B3 Error in perpendicular distance formula 
B4 Incorrect application of Pythagoras formula 
B5 Error in slope formula 
B6 Error in squaring 
B7 Error in factors 
B8 Equation of one circle only 

Slips (-1) 
S1 Arithmetic error 

Attempts (3, 2, 2 marks) 
A1 Mid point or slope \(pq\) 
A2 \(c\) expressed in terms of \(g\) 
A3 Radius only
QUESTION 2

Part (a) 10 (5, 5) marks  Att (2, 2)
Part (b) 20 (10, 10) marks  Att (3, 3)
Part (c) 20 (10, 10) marks  Att (3, 3)

Part (a) 10 (5, 5) marks  Att (2, 2)

\[ \mathbf{a} = 2 \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = -\mathbf{i} + 5 \mathbf{j}, \]

Find the unit vector in the direction of \( \mathbf{b} - \mathbf{a} \).

\[ \mathbf{b} - \mathbf{a} = \mathbf{b} - \mathbf{a} = -\mathbf{i} + 5 \mathbf{j} - 2 \mathbf{i} - \mathbf{j} = -3 \mathbf{i} + 4 \mathbf{j}. \]

\[ |\mathbf{b} - \mathbf{a}| = |-3 \mathbf{i} + 4 \mathbf{j}| = \sqrt{9 + 16} = 5. \]

Unit vector \[ \frac{\mathbf{b} - \mathbf{a}}{|\mathbf{b} - \mathbf{a}|} = \frac{-3 \mathbf{i} + 4 \mathbf{j}}{5} = -\frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}. \]

Blunders (-3)
B1 Error in \( \mathbf{b} - \mathbf{a} \)
B2 Error in formula for norm of vector
B3 Answer not expressed in correct form

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 Norm formula with some substitution
A2 \( \mathbf{b} - \mathbf{a} \) and stops

Part (b) 20 (10, 10) marks  Att (3, 3)

(b) In the triangle \( abc \), \( p \) is a point on the side \( [bc] \).

The point \( q \) lies outside the triangle such that \( \mathbf{pq} = \mathbf{pb} + \mathbf{pe} - \mathbf{pa} \).

(i) Express \( \mathbf{q} \) in terms of \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \).

(ii) Hence show that \( abqc \) is a parallelogram.
(b) (i) 10 marks Att 3

2 (b) (i)
\[ \vec{pq} = \vec{pb} + \vec{pc} - \vec{pa} \Rightarrow \vec{q} - \vec{p} = \vec{b} - \vec{p} + \vec{c} - \vec{a} + \vec{p}. \]
\[ \therefore \vec{q} = \vec{b} + \vec{c} - \vec{a}. \]

Blunders (-3)
B1 \( \vec{pq} \) or equivalent expressed incorrectly

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 \( \vec{pq} \) or equivalent expressed correctly

(b) (ii) 10 marks Att 3

2 (b) (ii)

By part (i): \( \vec{q} = \vec{b} + \vec{c} - \vec{a} \Rightarrow \vec{q} - \vec{b} = \vec{c} - \vec{a} \Rightarrow bq = ac. \)
\[ \therefore abqc \text{ is a parallelogram.} \]

Blunders (-3)
B1 \( \vec{c} - \vec{a} \neq \vec{ac} \)
B2 \( \vec{q} - \vec{b} \neq \vec{bq} \)
B3 No conclusion or incorrect conclusion

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 \( \vec{q} - \vec{b} = \vec{c} - \vec{a} \)

Part (c) 20 (10, 10) marks Att (3, 3)

(c) (i) \( \vec{p} = 12 \hat{i} + 5 \hat{j} \) and \( \vec{q} = 3 \hat{i} + 4 \hat{j} \).
Find the value of the scalar \( k \) such that
\[ k \left| \vec{p} - \vec{q} \right| = \left| \vec{p} \right| - \left| \vec{q} \right|. \]

(ii) Prove that for all vectors \( \vec{r} \) and \( \vec{s} \)
\[ \left( \vec{r} - \vec{s} \right) \perp = \vec{r}^\perp - \vec{s}^\perp. \]
Part (c) (i)  
10 marks  

\[ k \left| \vec{p} - \vec{q} \right| = \left| \vec{p} \right| - \left| \vec{q} \right| \Rightarrow k \left| -5 \vec{i} + 12 \vec{j} - 3 \vec{i} - 4 \vec{j} \right| = \left| -5 \vec{i} + 12 \vec{j} - 3 \vec{i} + 4 \vec{j} \right|. \]

\[ \therefore \left| -8 \vec{i} + 8 \vec{j} \right| = 13 - 5 \Rightarrow \sqrt{128k} = 8 \Rightarrow 8\sqrt{2}k = 8 \Rightarrow k = \frac{1}{\sqrt{2}} \Rightarrow k = \frac{\sqrt{2}}{2}. \]

Blunders (-3)
B1 \( \vec{p} \) incorrect
B2 Error in formula for norm of vector
B3 \( k \) not in surd form

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Norm of \( \vec{q} \)
A2 \( \vec{p} \) only

Part (c) (ii)  
10 marks  

Let \( \vec{r} = a \vec{i} + b \vec{j} \) and \( \vec{s} = c \vec{i} + d \vec{j} \). \( \therefore \vec{r} - \vec{s} = (a-c)\vec{i} + (b-d)\vec{j} \).

\[ \left( \vec{r} - \vec{s} \right) = -(b-d)\vec{i} + (a-c)\vec{j} \]

\[ \left( \vec{r} - \vec{s} \right) \perp \left( \vec{r} - \vec{s} \right) = -b\vec{i} + a\vec{j} - \left( -d\vec{i} + c\vec{j} \right) = -(b-d)\vec{i} + (a-c)\vec{j} = \left( \vec{r} - \vec{s} \right) \perp. \]

Blunders (-3)
B1 \( \vec{r} \) incorrect
B2 No conclusion or incorrect conclusion

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 One related perpendicular correct
A2 \( \vec{r} - \vec{s} \) expressed in terms of \( \vec{i} \) and \( \vec{j} \)
A3 Numerical values for \( \vec{r} \) and \( \vec{s} \) fully worked out ‘correctly’.
QUESTION 3

Part (a) 10 marks Att 3

3 (a) Find the equation of the line that contains the point (1, 0) and passes through the point of intersection of the lines $2x - y + 6 = 0$ and $10x + 3y - 2 = 0$.

\[
\begin{align*}
6x - 3y + 18 &= 0 \\
10x + 3y - 2 &= 0 \\
16x + 16 &= 0 \quad \Rightarrow \quad x = -1 \quad \text{and} \quad y = 4. \\
(1, 0) \quad \text{and} \quad (-1, 4) \quad \Rightarrow \quad m = \frac{0 - 4}{1+1} = -2. \\
\therefore \quad \text{Equation of line} : y - 0 = -2(x - 1) \quad \Rightarrow \quad 2x + y - 2 = 0.
\end{align*}
\]

Blunders (-3)
B1 Error in slope formula
B2 Error in equation of line formula

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 One co-ordinate of point of intersection
A2 $2x - y + 6 + \lambda (10x + 3y - 2) = 0$

Part (b) 20 (10, 10) marks Att (3, 3)

(b) (i) Prove that the measure of one of the angles between two lines with slopes $m_1$ and $m_2$ is given by

\[
\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}.
\]

(ii) Find the equations of the two lines that pass through the point (6, 1) and make an angle of $45^\circ$ with the line $x + 2y = 0$. 

Page 46 of 82
3 (b) (i)

Slope $L_1 = m_1$ and slope $L_2 = m_2$.

Let $\theta_1$ and $\theta_2$ be the positive angles made by $L_1$ and $L_2$ respectively with the positive sense of the $x$-axis.

Then $\tan \theta_1 = m_1$ and $\tan \theta_2 = m_2$.

**Case 1:** ($\theta_1 > \theta_2$)

$\theta = \theta_1 - \theta_2$,

$\tan \theta = \tan (\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}$

$\therefore \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$.

**Case 2:** ($\theta_1 < \theta_2$)

$\theta = \theta' + \theta_1 \Rightarrow \theta' = -(\theta_1 - \theta_2)$

$\tan \theta' = -\tan (\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}$

$= \frac{-m_1 - m_2}{1 + m_1 m_2}$.

In this case, the other angle between the lines is $\theta = 180^\circ - \theta'$, giving $\tan \theta = -\tan \theta'$.

* One case to be accepted for full marks

**Blunders (-3)**

B1 Error in expressing $\theta$ in terms of $\theta_1$ and $\theta_2$

B2 Error in expansion of $\tan (\theta_1 - \theta_2)$

**Slips (-1)**

S1 Arithmetic error

**Attempts (3 marks)**

A1 $\theta_1 = \theta + \theta_2$ and stops
Part (b) (ii) 10 marks

3 (b) (ii)

\[ x + 2y = 0 \] has slope \(-\frac{1}{2}\).

\[ \tan 45^\circ = \pm \frac{m_1 - m_2}{1 + m_1 m_2}, \text{ where } m_2 = -\frac{1}{2}. \]

\[ \therefore 1 = \pm \frac{m_1 + \frac{1}{2}}{1 - \frac{1}{2} m_1} \Rightarrow 2 - m_1 = \pm (2m_1 + 1) \]

\[ 2 - m_1 = 2m_1 + 1 \Rightarrow m_1 = \frac{1}{3} \text{ or } 2 - m_1 = -2m_1 - 1 \Rightarrow m_1 = -3. \]

\[ y - 1 = \frac{1}{3}(x - 6) \text{ and } y - 1 = -3(x - 6) \]

\[ x - 3y = 3 \text{ and } 3x + y = 19. \]

\textbf{Blunders (-3)}

B1 Error in slope
B2 Product of slopes \(\neq -1\)
B3 One equation only

\textbf{Slips (-1)}

S1 Arithmetic error

\textbf{Attempts (3 marks)}

A1 Slope of \(x + 2y = 0\)
A2 \(\tan 45^\circ = 1\)
### Part (c)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

<table>
<thead>
<tr>
<th>(c)</th>
<th>20 (5, 5, 5, 5) marks</th>
<th>Att (2, 2, 2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>( f ) is the transformation ((x, y) \rightarrow (x', y')), where (x' = -x + 2y) and (y' = 2x - y).</td>
<td></td>
</tr>
<tr>
<td>(i)</td>
<td>( L ) is the line (ax + by + c = 0). Prove that ( f(L) ) is a line.</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>The line (y = mx) is its own image under (f). Find the two possible values of (m).</td>
<td></td>
</tr>
</tbody>
</table>

#### (c) (i) \( x \) and \( y \) in terms of \( x' \) and \( y' \)

5 marks

| Substitution | 5 marks | Att 2 |
| Finish | 5 marks | Att 2 |

\[
\begin{align*}
  x' &= -x + 2y \\
  2y' &= 4x - 2y \\
  x' + 2y' &= 3x & \iff & x &= \frac{1}{3}(x' + 2y') \\
  y &= 2x - y' & \iff & y &= \frac{2}{3}(x' + 2y') - y' & \iff & y &= \frac{1}{3}(2x' + y') .
\end{align*}
\]

\( \therefore \) The inverse relation is a function and so \( f \) is clearly bijective \( \Pi_0 \rightarrow \Pi_0 \).

The set \( f(L) \) is the set of all points \((x', y')\) for which \((x, y) \in L\).

\[
ax + by + c = 0 \\
\iff \quad \frac{a}{3}(x' + 2y') + \frac{b}{3}(2x' + y') + c = 0 \\
\iff \quad (a + 2b)x' + (2a + b)y' + 3c = 0 .
\]

\( \therefore \) \( f(L) \) is a line, (since it consists of the set of all points satisfying an equation of the form \( px + qy + r = 0 \)).

#### OR

(c) (i) Apply \( f \) to vector form

5 marks

| Substitution | 5 marks | Att 2 |
| Finish | 5 marks | Att 2 |

\[
L \text{ is the set } \{ \vec{c} + \vec{m} \mid t \in \mathbb{R} \}, \text{ where } \vec{c} = \begin{pmatrix} 0 \\ -b \\ a \end{pmatrix} \text{ and } \vec{m} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} .
\]

\( \therefore \) \( f(L) \) is the set \( \{ f(\vec{c} + \vec{m}) \mid t \in \mathbb{R} \} = \{ (f(\vec{c}) + tf(\vec{m})) \mid t \in \mathbb{R} \} \), since \( f \) is linear.

This is a line, since \( f(\vec{m}) \neq \vec{0} \), (as \( \det(f) = -3 \neq 0 \) \( \rightarrow \) \( f \) is invertible).

---

**Blunder\(-3\)**

B1 \( f(L) \) not in the form \( px + qy + r = 0 \)

**Slip\(-1\)**

S1 Arithmetic error

**Attempts (2,2,2 marks)**

A1 Effort at \( x \) or \( y \) expressed in terms of \( x' \) and \( y' \)
(c) (ii) \[ 5 \text{ marks} \]

\[
(1, m) \in y = mx \quad \text{and} \quad f(1, m) = (-1 + 2m, 2 - m), \quad f(0, 0) = (0, 0).
\]

\[
\therefore \frac{m}{1} = \frac{2 - m}{-1 + 2m} \quad \text{as slope of line and slope of image line are equal.}
\]

\[
\therefore -m + 2m^2 = 2 - m \quad \Rightarrow \quad 2m^2 = 2 \quad \Rightarrow \quad m = \pm 1.
\]

OR

(c) (ii) \[ 5 \text{ marks} \]

\[
y = mx \quad \Leftrightarrow \quad mx - y + 0 = 0, \quad \text{so} \quad a = m, \quad b = -1, \quad c = 0.
\]

So, from part (i), the image is \((m - 2)x' + (2m - 1)y' + 0 = 0\)

Rearrange: \[ y' = \frac{-m + 2}{2m - 1}x' \]

This is the same line as \(y = mx\), so \[ \frac{-m + 2}{2m - 1} = m \].

\[
\therefore -m + 2m^2 = 2 - m \quad \Rightarrow \quad 2m^2 = 2 \quad \Rightarrow \quad m = \pm 1.
\]

Blunders (-3)

B1 Error in \(f(1, m)\) or equivalent

B2 One value of \(m\) only

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Correct image of any point

A2 Equation of \(y = mx\) under \(f\)
**QUESTION 4**

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (15, 5) marks</td>
<td>Att (5, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (10, 5, 5) marks</td>
<td>Att (3, 2, 2)</td>
</tr>
</tbody>
</table>

**Part (a) 10 marks**

Show \((\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 = 2\)

4 (a) 

\[
(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 = \cos^2 \theta + 2\cos \theta \sin \theta + \sin^2 \theta + \cos^2 \theta - 2\cos \theta \sin \theta + \sin^2 \theta \\
= 2(\cos^2 \theta + \sin^2 \theta) = 2.
\]

**Blunders (-3)**

B1 Error in squaring
B2 \(\cos^2 \theta + \sin^2 \theta \neq 1\)
B3 Incorrect conclusion

**Slips (-1)**

S1 Arithmetic error

**Attempts (3 marks)**

A1 One expansion correct
A2 Verification fully correct

**Part (b) 20 (15, 5) marks**

(b) The lengths of the sides of a triangle are 21, 17 and 10.

The smallest angle in the triangle is \(A\).

(i) Show that \(\cos A = \frac{15}{17}\).

(ii) Without evaluating \(A\), find \(\tan \frac{A}{2}\).

(b) (i) 15 marks

The smallest angle is opposite the smallest side, so take \(a = 10\).

\[
\cos A = \frac{b^2 + c^2 - a^2}{2bc} \\
\cos A = \frac{21^2 + 17^2 - 10^2}{2(21)(17)} = \frac{441 + 289 - 100}{714} = \frac{630}{714} = \frac{15}{17}.
\]

**Blunders (-3)**

B1 Error in Cosine formula
B2 Error in substitution

**Slips (-1)**

S1 Arithmetic error

**Attempts (5 marks)**

A1 Some values substituted into Cosine formula
A2 Cosine \(A\) expressed in terms of the sides of triangle
(b) (ii) 5 marks

\[
\cos A = \frac{15}{17} = \frac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)} \Rightarrow 15 + 15\tan^2\left(\frac{A}{2}\right) = 17 - 17\tan^2\left(\frac{A}{2}\right).
\]

\[
\Rightarrow 32\tan^2\left(\frac{A}{2}\right) = 2 \Rightarrow \tan^2\left(\frac{A}{2}\right) = \frac{1}{16} \Rightarrow \tan \frac{A}{2} = \frac{1}{4}, \text{ (positive, since } 0 < \frac{A}{2} < 90^\circ).\]

OR

(b) (ii) 5 marks

\[
\cos^2 A = \frac{1}{2}(1 + \cos 2A)
\]

\[
\cos^2 \frac{A}{2} = \frac{1}{2}(1 + \cos A)
\]

\[
= \frac{1}{2} \left(1 + \frac{15}{17}\right) = \frac{16}{17}
\]

\[
\cos \frac{A}{2} = \pm \frac{4}{\sqrt{17}}.
\]

But \(0 < \frac{A}{2} < \frac{\pi}{2}\), so \(\cos \frac{A}{2} = \frac{4}{\sqrt{17}} = \frac{\text{adj}}{\text{hyp}} \text{ in a right angled triangle}
\]

\[
\left(\sqrt{17}\right)^2 = 4^2 + \text{opp}^2 \Rightarrow \text{opp} = 1
\]

\[
\Rightarrow \tan \frac{A}{2} = \frac{1}{4}
\]

Blunders (-3)
B1 Error in formula
B2 \(\tan \frac{A}{2}\) negative

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 \(\tan \frac{A}{2}\) substituted correctly
(c) The bisector of \( \angle qpr \) meets \( qr \) at \( s \).

\[ \angle qpr = 2\theta, \quad |pq| = x, \]
\[ |pr| = y \text{ and } |ps| = k. \]

(i) Find the area of the triangle \( pqs \) in terms of \( x, k \) and \( \theta \).

(ii) Show that \( k = \frac{2xy \cos \theta}{x + y} \).
QUESTION 5

Part (a) 10 marks Att 3
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

Part (a) 10 marks Att 3

5 (a) Find all the solutions of the equation \( \cos^2 x - \cos x = 0 \), where \( 0^\circ \leq x \leq 180^\circ \).

5 (a) \[
\cos^2 x - \cos x = 0 \Rightarrow \cos x(\cos x - 1) = 0
\]

\[
\cos x = 0 \Rightarrow x = 90^\circ \quad \text{or} \quad \cos x = 1 \Rightarrow x = 0^\circ
\]

Solution is \{ 0^\circ, 90^\circ \}

Blunders (-3)
B1 Incorrect factors
B2 Each incorrect value
B3 Each omitted value or ‘extra’ value

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 \( \cos x = 0 \) or \( \cos x - 1 = 0 \)

Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(b) The function \( f : x \rightarrow \sin^{-1} x \) is defined for \(-1 \leq x \leq 1\).

(i) Copy and complete the table of values of \( f \) below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>(-\frac{\sqrt{3}}{2})</th>
<th>(-\frac{1}{2})</th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>(\frac{\sqrt{3}}{2})</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(-\frac{\pi}{6})</td>
<td></td>
</tr>
</tbody>
</table>

(ii) Draw the graph of \( y = f(x) \) on graph paper, noting that \( \frac{\sqrt{3}}{2} \approx 0.87 \).

Scale the \( y \)-axis in terms of \( \pi \).

(iii) State, with reason, whether each of the following statements is true.

A: “If \( \sin x_1 = \sin x_2 \), then \( x_1 = x_2 \).”

B: “If \( \sin^{-1} x_1 = \sin^{-1} x_2 \), then \( x_1 = x_2 \).”
(b) (i) 5 marks  

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>-1</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>f(x)</td>
<td>$-\frac{\pi}{2}$</td>
<td>$-\frac{\pi}{3}$</td>
<td>$-\frac{\pi}{6}$</td>
<td>0</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{3}$</td>
</tr>
</tbody>
</table>

**Slips (-1)**

S1 Each incorrect entry to max of 3

**Attempts (2 marks)**

A1 One correct entry

(b) (ii) 5 marks

**Blunders (-3)**

B1 $x$ axis scaled in terms of $\pi$ (instead of $y$ axis)
B2 Error in scales
B3 Not joining points

**Slips (-1)**

S1 Each incorrect plot to max of 3.

**Attempts (2 marks)**

A1 Axes with some correct scale
A2 One point correctly indicated
(b) (iii) A  5 marks
B  5 marks

5 (b) (iii)

A is False:
For example, \( \sin 150^\circ = \sin 30^\circ \), while \( 150^\circ \neq 30^\circ \)
or
A horizontal line can cut the graph of \( y = \sin(x) \) more than once.

B is True:
A horizontal line can’t cut the graph of \( y = \sin^{-1} x \) more than once.
or
\( \sin^{-1} \) is strictly increasing on its domain
or
\( \sin^{-1} \) is bijective

Blunders (-3)
B1 Correct answer no reason given
B2 Correct answer, incorrect reason

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 \( \sin 150^\circ = \sin 30^\circ \) or equivalent
A rectangular block of cheese measures 8 cm \times 8 cm \times 4 cm.

One corner is cut away from the block, in such a way that three of the edges are cut through their midpoints \(a\), \(b\) and \(c\).

Find the area of the triangular face \(abc\) created by the cut.

\[
\begin{align*}
|ab|^2 &= 4^2 + 2^2 \
|ac|^2 &= 4^2 + 4^2 \\
\cos \angle abc &= \frac{|ab|^2 + |bc|^2 - |ac|^2}{2|ab||bc|} = \frac{20 + 20 - 32}{40} = \frac{8}{40} = \frac{1}{5}.
\end{align*}
\]

\[
\therefore \sin \angle abc = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}
\]

\[
\Rightarrow \text{area triangle } abc = \frac{1}{2} |ab||bc|\sin \angle abc = \frac{1}{2} (\sqrt{20})(\sqrt{20})\frac{2\sqrt{6}}{5} = 4\sqrt{6} \text{ cm}^2.
\]

OR

\[
\begin{align*}
|ab| &= \sqrt{20} \\
|ac| &= \sqrt{32} \\
h &= 12 \\
\Rightarrow h &= 2\sqrt{3}
\end{align*}
\]

\[
\text{Area } \Delta abc = \frac{1}{2} \cdot 2\sqrt{3} \cdot 2\sqrt{3} \cdot \sqrt{96} = 4\sqrt{6} \text{ cm}^2
\]

Blunders (-3)

B1 Pythagoras incorrect
B2 Incorrect substitution into cosine formula
B3 Incorrect area formula
B4 Area not calculated
B5 \(\sin A\) incorrectly evaluated from \(\cos A\)
Slips (-1)
S1 Arithmetic error
S2 Units omitted

Attempts (2, 2, 2, 2 marks)
A1 Incorrect use of Pythagoras
A2 Some substitution into cosine formula
### QUESTION 6

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5) marks</td>
<td>Att (2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 10) marks</td>
<td>Att (2, 2, 3)</td>
</tr>
</tbody>
</table>

#### Part (a) 10 (5, 5) marks  Att (2, 2)

(a) A student taking a literature course has to read three novels from a list of ten novels.

   (i) How many different selections of three novels are possible?

   (ii) Two of the ten novels are by the same author. How many selections are possible if the student wishes to choose three novels by different authors?

#### (a) (i) 5 marks  Att 2

\[
\text{Number of selections } = \binom{10}{3} = 120
\]

**Blunders (-3)**

- **B1** $10 \times 9 \times 8$

**Attempts (2 marks)**

- **A1** \( \binom{x}{10}, x \in \mathbb{N} \)

#### (a) (ii) 5 marks  Att 2

\[
\text{Number of selections } = \binom{2}{1} \times \binom{8}{2} + \binom{8}{3} = 56 + 56 = 112.
\]

\[
\text{or } \binom{10}{3} - \binom{8}{1} = 120 - 8 = 112
\]

\[
\text{or } \binom{8}{1} + \binom{8}{2} = 84 + 28 = 112
\]

**Blunders (-3)**

- **B1** \( \binom{2}{1} \) or equivalent missing
- **B2** \( \binom{2}{1} \times \binom{8}{2} \times \binom{8}{3} \)
- **B3** \( \binom{2}{1} \times \binom{8}{2} \)
- **B4** \( \binom{8}{1} \) or \( \binom{8}{3} \)

**Attempts (2 marks)**

- **A1** \( \binom{8}{2} \) or \( \binom{8}{3} \) or \( \binom{8}{1} \)
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(b) (i) In how many different ways can eight people be seated in a row?

(ii) Three girls and five boys sit in a row, arranged at random. Find the probability that the three girls are seated together.

(iii) Three girls and \( n \) boys sit in a row, arranged at random. If the probability that the three girls are seated together is \( \frac{1}{35} \), find the value of \( n \).

### Part (b) (i) 5 marks Att 2

6 (b) (i)

Number of ways = 8!.

**Blunders (-3)**
- B1 \(^8C_3 \) or \(8^8\)

**Slips (-1)**
- S1 Arithmetic error

**Attempts (2 marks)**
- A1 \(8 + 7 + 6 + 5 + 4 + 3 + 2 + 1\)

### Part (b) (ii) 5 marks Att 2

6 (b) (ii)

Number of possible arrangements = 8!.

Number of favourable arrangements = 6! \times 3!.

Probability = \(\frac{6! \times 3!}{8!} = \frac{6}{56} = \frac{3}{28}\).

**Blunders (-3)**
- B1 Incorrect number of possible outcomes
- B2 Incorrect number of favourable outcomes (e.g. 5! \times 3!)
- B3 6! + 3!
- B4 6! \times 3
- B5 No divisor

**Slips (-1)**
- S1 Arithmetic error

**Attempts (2 marks)**
- A1 Correct number of favourable outcomes
- A2 Correct number of possible outcomes
6 (b) (iii)

Number of possible arrangements = \((n + 3)!\).
Number of favourable arrangements = \((n + 1)! \times 3!\).

\[
\therefore \text{Probability} = \frac{(n + 1)! \times 3!}{(n + 3)!} = \frac{6}{(n + 3)(n + 2)}.
\]

\[
\therefore \frac{6}{(n + 3)(n + 2)} = \frac{1}{35}
\]

\[
\therefore n^2 + 5n + 6 = 210 \Rightarrow n^2 + 5n - 204 = 0.
\]

\[
(n - 12)(n + 17) = 0 \quad \Rightarrow \quad n = 12, \text{ as } n \neq -17.
\]

Solution is \(n = 12\).
Part (c) 20 (5, 5, 10) marks  Att (2, 2, 3)

(c) 

\[ x \text{ and } y \text{ are randomly selected integers with } 1 \leq x \leq 10 \text{ and } 1 \leq y \leq 10. \]

\[ p \text{ is the point with coordinates } (x, 0) \text{ and } q \text{ is the point with coordinates } (0, y). \]

Find the probability that

(i) the slope of \( pq \) is equal to \(-1\)

(ii) the slope of \( pq \) is greater than \(-1\)

(iii) the length of \([pq]\) is less than or equal to 5.

![Diagram showing the points p and q]

\[
\begin{array}{c|cccccccccc}
\text{x} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
2 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
3 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
4 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
5 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
6 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
7 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
8 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
9 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
10 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array}
\]

OR

Part (c) (i) 5 marks  Att 2

\[
\text{Slope } pq = \frac{y - 0}{0 - x} = -\frac{y}{x} \text{ and } -\frac{y}{x} = -1, \text{ when } x = y. \\
\therefore \text{ Number of favourable outcomes is 10.} \\
\text{Number of possible outcomes is } 10 \times 10 = 100. \\
\therefore \text{ Probability } = \frac{10}{100} = \frac{1}{10}. 
\]

Blunders (-3)
B1 \( y \neq x \) implied
B2 Incorrect number of possible outcomes
B3 Incorrect number of favourable outcomes

Slips (-1)
S1 Arithmetic error
Attempts (2 marks)
A1 Listing some favourable outcomes
A2 Listing total number of outcomes

Part (c) (ii) 5 marks Att 2

\[
pq = \frac{y - 0}{0 - x} = -\frac{y}{x}, \quad \frac{y}{x} < -1 \Rightarrow \frac{y}{x} < 1 \Rightarrow y < x.
\]
\[
\therefore \text{Number of favourable outcomes is } 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45.
\]
Number of possible outcomes is \(10 \times 10 = 100\).
\[
\therefore \text{Probability } = \frac{45}{100} = \frac{9}{20}.
\]

Blunders (-3)
B1 \(y < x\) not implied
B2 Incorrect number of possible outcomes
B3 Incorrect number of favourable outcomes

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Listing favourable outcomes
A2 Listing total number of outcomes
A3 Drawing a grid with some relevant items

Part (c) (iii) 10 marks Att 3

\[
|pq| = \sqrt{x^2 + y^2}. \quad |pq| \leq 5 \Rightarrow x^2 + y^2 \leq 25.
\]
\[
\therefore \text{Favourable outcomes are }\{1, 2, 3, 4\} \text{ and }\{1, 2, 3, 4\} \text{ but with } x = 4 \text{ and } y = 4 \text{ not included.}
\]
\[
\therefore \text{Number of favourable outcomes is } (4 \times 4) - 1 = 15.\]

Number of possible outcomes is \(10 \times 10 = 100\).
\[
\therefore \text{Probability } = \frac{15}{100} = \frac{3}{20}.
\]

Blunders (-3)
B1 \(x^2 + y^2 \leq 25\) or equivalent not implied
B2 Incorrect number of possible outcomes
B3 Incorrect number of favourable outcomes

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Listing some favourable outcomes
A2 Listing total number of outcomes
A3 Some use of Pythagoras
## QUESTION 7

### Part (a) 10 marks

7 (a) The prices of four food items in a shopping basket are €3, €5, €1 and €6.
Find the weighted mean price of these items using the weights 2, 3, 4 and 1 respectively.

### Part (a) 10 marks

**Weighted mean**

\[
\frac{2(3) + 3(5) + 4(1) + 1(6)}{10} = \frac{31}{10} = €3.10.
\]

### Blunders (-3)

B1 Sum of weights incorrect
B2 Incorrect denominator
B3 \(x + w\) instead of \(xw\) for each term

### Slips (-1)

S1 Arithmetic error
S2 Omission of currency symbol

### Attempts (3 marks)

A1 Sum of weights

### Part (b) 20 (5, 5, 5, 5) marks

(b) (i) Solve the difference equation \(u_{n+2} - 6u_{n+1} + 5u_n = 0\), where \(n \geq 1\),
given that \(u_1 = 0\) and \(u_2 = 20\).

(ii) Find an expression in \(n\) for the sum of the terms
\(u_1 + u_2 + u_3 + \ldots + u_n\).

### Part (b) (i) Characteristic Equation 5 marks

\[
u_n = p(\alpha)^n + q(\beta)^n = p(1)^n + q(5)^n \Rightarrow u_n = p + q(5)^n.
\]

### Part (b) (i) Finish 5 marks

\[
u_1 = p + 5q = 0 \quad \text{and} \quad u_2 = p + 25q = 20.
\]

\[
\therefore 20q = 20 \Rightarrow q = 1 \quad \text{and hence} \quad p = -5.
\]

\[
\therefore u_n = -5 + 5^n.
\]
Blunders (-3)
B1 Error in setting up quadratic
B2 Error in solving quadratic
B3 Error in general term
B4 Error in finding p and q

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 Substitution into quadratic formula
A2 Attempt at finding p or q

Part (b) (ii) 5 marks  

\[
\begin{align*}
    u_1 + u_2 + u_3 + \ldots + u_n &= \sum_{n=1}^{n} u_n = -5n + \sum_{n=1}^{n} 5^n \\
    &= -5n + \frac{5(5^n - 1)}{5 - 1} = -5n + \frac{5}{4} (5^n - 1)
\end{align*}
\]

Blunders (-3)
B1 Error in forming geometric series
B2 Error in sum of geometric series
B3 Mishandling -5

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 Using formula for sum to infinity of G.P.
A2 Correct formula with some substitution
A3 Listing at least 3 consecutive terms correctly
Part (c) 20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

(c) The two numbers \(a\) and \(b\) have mean \(\bar{x}\) and standard deviation \(\sigma_1\).

The three numbers \(c\), \(d\) and \(e\) have mean \(\bar{x}\) and standard deviation \(\sigma_2\).

Find the standard deviation of the five numbers \(a\), \(b\), \(c\), \(d\) and \(e\) in terms of \(\sigma_1\) and \(\sigma_2\).

### Part (c) Expressions for \(\sigma_1\) and \(\sigma_2\) 5 marks  Att 2

Both \(\bar{x}\) for \((a, b)\) and \((c, d, e)\) 5 marks  Att 2

Mean \(a, b, c, d, e\) 5 marks  Att 2

Finish 5 marks  Att 2

\[
\begin{align*}
a + b &= \frac{a + b}{2} = \bar{x}, \\
c + d + e &= \frac{c + d + e}{3} = \bar{x}.
\end{align*}
\]

\[
\sigma_1 = \sqrt{\frac{(a - \bar{x})^2 + (b - \bar{x})^2}{2}}, \quad \sigma_2 = \sqrt{\frac{(c - \bar{x})^2 + (d - \bar{x})^2 + (e - \bar{x})^2}{3}}.
\]

Mean of \(a, b, c, d, e\) \(= \frac{a + b + c + d + e}{5} = \frac{2\bar{x} + 3\bar{x}}{5} = \bar{x}.
\]

Standard deviation of \(a, b, c, d, e\) \(= \sqrt{\frac{(a - \bar{x})^2 + (b - \bar{x})^2 + (c - \bar{x})^2 + (d - \bar{x})^2 + (e - \bar{x})^2}{5}}
\]
\[
= \sqrt{\frac{2\sigma_1^2 + 3\sigma_2^2}{5}}.
\]

OR

### Part (c) Expressions for \(\sigma_1\) and \(\sigma_2\) 5 marks  Att 2

Both \(\bar{x}\) (for \(a, b\) and \(c, d, e\)) 5 marks  Att 2

Mean \(a, b, c, d, e\) 5 marks  Att 2

Finish 5 marks  Att 2

\[
\begin{align*}
\sum \frac{x^2}{n} - (\bar{x})^2 &\Rightarrow \sigma_1^2 = \frac{a^2 + b^2}{2} - \bar{x}^2 \quad \text{and} \quad \sigma_2^2 = \frac{c^2 + d^2 + e^2}{3} - \bar{x}^2
\end{align*}
\]

But \(a + b = \frac{a + b}{2} = \bar{x}\) and \(c + d + e = \frac{c + d + e}{3} = \bar{x} \Rightarrow \frac{a + b + c + d + e}{5} = \frac{2\bar{x} + 3\bar{x}}{5} = \bar{x}
\]

\[
\begin{align*}
\sigma^2 &= \frac{a^2 + b^2 + c^2 + d^2 + e^2}{5} - \bar{x}^2 \\
&= \frac{a^2 + b^2 + c^2 + d^2 + e^2 - 5\bar{x}^2}{5} \\
&= \frac{a^2 + b^2 - 2\bar{x}^2 + c^2 + d^2 + e^2 - 3\bar{x}^2}{5} \\
&= \frac{2\sigma_1^2 + 3\sigma_2^2}{5}.
\end{align*}
\]

\(\therefore \sigma = \sqrt{\frac{2\sigma_1^2 + 3\sigma_2^2}{5}}
\]
Blunders (-3)
B1  Error in mean
B2  Error in standard deviation

Slips (-1)
S1  Arithmetic error

Attempts (2, 2, 2, 2 marks)
A1  Correct mean of $a$ and $b$
A2  One correct standard deviation
A3  Expression for mean of $a$, $b$, $c$, $d$, $e$
QUESTION 8

Part (a) 15 (5, 5, 5) marks Att (2, 2, 2)
Part (b) 20 (15, 5) marks Att (5, 2)
Part (c) 15 (5, 5, 5) marks Att (2, 2, 2)

Part (a) 15 (5, 5, 5) marks Att (2, 2, 2)

8. (a) Use integration by parts to find \( \int x e^{4x} \, dx \).

Part (a) Assign parts 5 marks Att 2
\( \frac{du}{dx} \) and \( v \) 5 marks Att 2
Finish 5 marks Att 2

\[
\int x e^{4x} \, dx = uv - \int v du, \quad \text{where} \quad u = x \Rightarrow du = dx \quad \text{and} \quad dv = e^{4x} \, dx \quad \Rightarrow \quad v = \frac{1}{4} e^{4x}.
\]

\[
\therefore \int x e^{4x} \, dx = \frac{1}{4} xe^{4x} - \int \frac{1}{4} e^{4x} \, dx = \frac{1}{4} xe^{4x} - \frac{1}{16} e^{4x} + c = \frac{e^{4x}}{16} (4x - 1) + c.
\]

Blunders (-3)
B1 Incorrect differentiation or integration
B2 Incorrect ‘parts’ formula

Slips (-1)
S1 Arithmetic error
S2 Omits constant of integration

Attempts (2,2,2 marks)
A1 One correct assigning to parts formula
A2 Correct differentiation or integration
Part (b) 20 (15, 5) marks Att (5, 2)

(b) (i) Derive the first four terms of the Maclaurin series for \( f(x) = \sqrt{1 + x} \).

(ii) Given that this series converges for \(-1 < x < 1\), use these four terms to find an approximation for \( \sqrt{17} \), as a fraction.

(b) (i) 15 marks Att 5

\[
\begin{align*}
8 \ (b) \ (i) & \quad f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots \\
& \quad f(x) = \sqrt{1 + x} \quad \Rightarrow \quad f(0) = 1. \\
& \quad f'(x) = \frac{1}{2}(1 + x)^{-\frac{1}{2}} \quad \Rightarrow \quad f'(0) = \frac{1}{2}. \\
& \quad f''(x) = -\frac{1}{4}(1 + x)^{-\frac{3}{2}} \quad \Rightarrow \quad f''(0) = -\frac{1}{4}. \\
& \quad f'''(x) = \frac{3}{8}(1 + x)^{-\frac{5}{2}} \quad \Rightarrow \quad f'''(0) = \frac{3}{8}. \\
\therefore \quad f(x) &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots
\end{align*}
\]

Blunders (-3)
B1 Incorrect differentiation
B2 Incorrect evaluation of \( f^{(n)}(0) \)
B3 Each term not derived
B4 Error in Maclaurin series

Slips(-1)
S1 Arithmetic error

Attempts (5 marks)
A1 Correct expansion for \( f(x) \) given but not derived
A2 \( f(0) \) correct
A3 A correct differentiation
A4 Any one correct term
(b) (ii) 5 marks At 2

8 (b) (ii)

\[
\sqrt{17} = \sqrt{16 + 1} = 4\sqrt{1 + \frac{1}{16}} = 4\sqrt{1 + \frac{x}{16}}, \text{ for } x = \frac{1}{16}.
\]

\[
\therefore \sqrt{17} = 4 \left[ 1 + \frac{1}{32} - \frac{1}{2048} + \frac{1}{65536} + \ldots \right] = 4 \left[ \frac{67553}{65536} \right] = 67553.16384
\]

Blunders (-3)
B1 Mishandling of \( \sqrt{16 + 1} \)
B2 Answer not in form \( \frac{a}{b}, \ a \in \mathbb{Z}, \ b \in \mathbb{Z} \)

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 17 as sum of 16 and 1 or 17 as sum of 9 and 8
A2 Answer in decimal form with relevant work

Part (c) 15 (5, 5, 5) marks At (2, 2, 2)

(c) The diagram shows a cylinder inscribed in a sphere.
The cylinder has height 2x and radius r.
The sphere has fixed radius a.

(i) Express r in terms of a and x.

(ii) Find, in terms of a, the maximum possible volume of the cylinder.

(c) (i) 5 marks At 2

8 (c) (i)

\[r^2 + x^2 = a^2\]

\[r^2 = a^2 - x^2\]

\[\Rightarrow r = \sqrt{a^2 - x^2}\]

Blunders (-3)
B1 Error in Pythagoras
B2 Incorrect side in triangle

Slips (-1)
S1 Arithmetic error

Attempts (2 marks)
A1 \( a^2 = r^2 + x^2 \)
(c)(ii) Volume in terms of $x$

Volume of cylinder $V = \pi r^2 h$.

\[ V = \pi (a^2 - x^2)2x = 2\pi a^2 x - 2\pi x^3. \]

\[ \frac{dV}{dh} = 2\pi a^2 - 6\pi x^2 = 0 \text{ for max or min.} \Rightarrow x = \frac{a}{\sqrt{3}}. \]

\[ \frac{d^2V}{dh^2} = -12\pi x < 0, \text{ for } x = \frac{a}{\sqrt{3}} \]

\[ \Rightarrow \text{maximum volume at } x = \frac{a}{\sqrt{3}}. \]

\[ \therefore V = \pi \left( a^2 - \frac{a^2}{3} \right) \frac{2a}{\sqrt{3}} = \frac{4\pi a^3}{3\sqrt{3}} = \frac{4\sqrt{3}\pi a^3}{9}. \]

* \[ \frac{d^2V}{dh^2} \] not required

Blunders (-3)

B1 Error in differentiation
B2 Error in finding $x$
B3 Error in indices

Slips (-1)

S1 Arithmetic error

Attempts (2, 2 marks)

A1 Some part of correct substitution into volume
A2 Some correct differentiation
A3 $\frac{dV}{dx} = 0$ indicated for max or min
QUESTION 9

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

Part (a) 10 marks Att 3

9. (a) $A$ and $B$ are independent events such that $P(A) = 0.25$ and $P(A \cup B) = 0.55$.

Find $P(B)$.

Part (a) Apply independence rule 5 marks Att 2
Finish 5 marks Att 2

9 (a)

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B).
\]
\[
\therefore P(A \cup B) = P(A) + P(B) - P(A)P(B).
\]
\[
\therefore 0.55 = 0.25 + P(B) - (0.25)P(B) \Rightarrow 0.75P(B) = 0.3.
\]
\[
\therefore P(B) = 0.4.
\]

OR

A and B independent $\iff$ $A'$ and $B'$ independent.

\[
P(A')P(B') = P(A' \cap B') = P((A \cup B)')
\]
\[
0.75P(B') = 0.45
\]
\[
P(B') = 0.6
\]
\[
P(B) = 0.4
\]

OR

\[
P(A \cap B) = P(A)P(B)
\]
\[
x = (0.25)(0.3 + x)
\]
\[
= 0.075 + 0.25x
\]
\[
0.75x = 0.075
\]
\[
x = 0.1
\]
\[
P(B) = 0.4
\]

Blunders (-3)
B1 $P(A \cap B) \neq P(A).P(B)$
B2 $P(A \cup B) = P(A) + P(B) + P(A).P(B)$

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Page 72 of 82
Part (b)  20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

(b) A person plays a game that involves throwing five hoops at a peg. The following table gives the probability distribution for the number of hoops that land on the peg.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(x)</td>
<td>0.01</td>
<td>0.08</td>
<td>0.23</td>
<td>0.34</td>
<td>0.26</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Find the mean and the standard deviation of the distribution.

Part (b) Mean  5 marks  Att 2
Deviations  5 marks  Att 2
\((x - \bar{x})^2 \cdot P(x)\) expressed  5 marks  Att 2
Finish  5 marks  Att 2

9 (b)

Mean = \(\bar{x} = \sum_{x=0}^{5} xP(x) = 0.08 + 0.46 + 1.02 + 1.04 + 0.4 = 3.\)

Standard deviation = \(\sigma = \sqrt{\sum_{x=0}^{5} (x - \bar{x})^2 \cdot P(x)}\).

\[= \sqrt{(0-3)^2(0.01)+(1-3)^2(0.08)+(2-3)^2(0.23)+(3-3)^2(0.34)+(4-3)^2(0.26)+(5-3)^2(0.08)}\]

\[\therefore \sigma = \sqrt{0.09 + 0.32 + 0.23 + 0 + 0.26 + 0.32} = \sqrt{1.22}.\]

Blunders (-3)
B1 \(\sum P(x)\) incorrect
B2 \(\sum x\) denominator for mean
B3 Use of \(\sum (x + P(x))\)
B4 Mishandles deviation
B5 Incorrect standard deviation formula

Slips (-1)
S1 Arithmetic error

Attempts (2,2,2,2 marks)
A1 Any correct \(x \cdot P(x)\)
A2 Correct formula with some substitution
A3 Any correct deviation
A coin is slightly bent and is thought to favour heads. Accordingly, it is tossed 100 times to test the null hypothesis that it is fair against the alternative hypothesis that it favours heads. In this experiment, 55 heads are observed.

(i) Show that this result is not significant at the 5% level.

(ii) How many times would the coin have to be tossed in an experiment in order that an observation of 55% heads would be regarded as significant at the 5% level?

\[
\begin{align*}
\sigma & = \sqrt{npq} = 5. \\
\therefore & \text{ The observed result is not significant at the 5% level.}
\end{align*}
\]

Blunders (-3)

- Incorrect value of \( p \) or of \( q \)
- Incorrect formula for mean
- Incorrect formula for standard deviation
- Error in standard units
- Two tailed test
- Misreads tables
- Incorrect conclusion

Slips (-1)

- Arithmetic error

Attempts (2, 2 marks)

- Correct value for \( p \) or \( q \)
- Correct formula for mean with some substitution
- Correct formula for standard deviation with some substitution
- Correct expression for standard units with some substitution
9 (c) (ii) Value of $z$

\[
\bar{x} = np = \frac{n}{2}, \quad \sigma = \sqrt{npq} = \frac{\sqrt{n}}{2}.
\]

\[
z = \frac{55n - \frac{n}{2}}{\frac{\sqrt{n}}{2}} = \frac{n}{\frac{n}{10}} = \frac{\sqrt{n}}{10}.
\]

\[
\frac{\sqrt{n}}{10} > 1.645 \Rightarrow \sqrt{n} > 16.45 \Rightarrow n > 270.6.
\]

\[
\therefore \text{271 times}
\]

**Blunders (-3)**

B1 Incorrect value of $p$ or of $q$
B2 Incorrect formula for mean
B3 Incorrect formula for standard deviation
B4 Error in standard units
B5 Mishandles 55%
B6 Incorrect conclusion

**Slips (-1)**

S1 Arithmetic error
S2 Stops at $n > 270.6$

**Attempts (2 marks)**

A1 Correct value for $p$ or $q$
A2 Some substitution for standard units
QUESTION 10

Part (a) 10 marks Att 3

10. (a) If \(a\) is the permutation \[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 2
\end{pmatrix}
\]
, find \(a \circ a\).

\[
a \circ a = \begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 2
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3
\end{pmatrix}
\]

Blunders (-3)
B1 Each incorrect element (max. of 2)

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 Permutation incomplete
A2 One element correct with another repeated

Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(b) The set \(\{1, 2, 4, 5, 7, 8\}\) is a group under multiplication modulo 9.

(i) Draw up a Cayley table for the group.

(ii) Find a generator of the group.

(iii) Hence, or otherwise, find a subgroup of order 2 and a subgroup of order 3.

\[
\begin{array}{cccccc}
\times_{\text{mod } 9} & 1 & 2 & 4 & 5 & 7 & 8 \\
1 & 1 & 2 & 4 & 5 & 7 & 8 \\
2 & 2 & 4 & 8 & 1 & 5 & 7 \\
4 & 4 & 8 & 7 & 2 & 1 & 5 \\
5 & 5 & 1 & 2 & 7 & 8 & 4 \\
7 & 7 & 5 & 1 & 8 & 4 & 2 \\
8 & 8 & 7 & 5 & 4 & 2 & 1 \\
\end{array}
\]
Blunders (-3)
B1  Not closed

Slips (-1)
S1  Arithmetic error
S2  Each incorrect entry to max of 3

Attempts (2 marks)
A1  Incomplete table

(b) (ii)  5 marks  Att 2

<table>
<thead>
<tr>
<th>10 (b) (ii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 is a generator.  ( 5^1 = 5, \ 5^2 = 7, \ 5^3 = 8, \ 5^4 = 4, \ 5^5 = 2, \ 5^6 = 1. )</td>
</tr>
<tr>
<td>or 2 is also a generator  ( 2^1 = 2, \ 2^2 = 4, \ 2^3 = 8, \ 2^4 = 7, \ 2^5 = 5, \ 2^6 = 1 )</td>
</tr>
</tbody>
</table>

Blunders (-3)
B1  Error in verifying generator
B2  Identity or other element not shown in terms of generator

Attempts (2 marks)
A1  Generator identified but not demonstrated for any element
A2  Attempts to establish a generator

Part (b) (iii)  10 (5, 5) marks  Att (2, 2)

<table>
<thead>
<tr>
<th>10 (b) (iii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgroup of order 2 is {1, 8}.</td>
</tr>
<tr>
<td>Subgroup of order 3 is {1, 4, 7}</td>
</tr>
</tbody>
</table>

Blunders (-3)
B1  Incorrect element in subgroup

Slips (-1)
S1  Arithmetic error

Attempts (2, 2 marks)
A1  Identity only correct element
Part (c) \[20(5, 5, 5, 5)\] marks \[\text{Att}(2, 2, 2, 2)\]

(c) \((G, \circ)\) and \((H, \ast)\) are two groups with identities \(e_G\) and \(e_H\) respectively.

If \(\phi : G \to H\) is an isomorphism, prove that

(i) \(\phi(e_G) = e_H\).

(ii) \(\phi(x^{-1}) = [\phi(x)]^{-1}\), for all \(x \in G\).

(c) (i) Establish correspondence \[5\] marks \[\text{Att} 2\]
Finish \[5\] marks \[\text{Att} 2\]

10 (c) (i)

Let \(x \in G\).
\[
\phi(x \circ e_G) = \phi(x) * \phi(e_G),
\]
because of isomorphism.
\[
\phi(x) = \phi(x) * \phi(e_G), \text{ as } x \circ e_G = x.
\]
\[
\Rightarrow \phi(e_G) \text{ is the identity in } (H, \ast).
\]
\[
\therefore \phi(e_G) = e_H.
\]

Blunders (-3)
B1 Error in setting up correspondence in operators
B2 No conclusion

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 \(x \circ e_G = x\)

(c) (ii) Establish correspondence \[5\] marks \[\text{Att} 2\]
Finish \[5\] marks \[\text{Att} 2\]

10 (c) (ii)

\[
\phi(\circ x^{-1}) = \phi(x) * \phi(x^{-1})
\]
\[
\therefore \phi(e_G) = \phi(x) * \phi(x^{-1}) \Rightarrow \phi(x) \text{ and } \phi(x^{-1}) \text{ are inverses.}
\]
\[
\therefore \phi(x^{-1}) = [\phi(x)]^{-1}.
\]

Blunders (-3)
B1 \(x \circ x^{-1} \neq e_G\) or equivalent
B2 Not indicating \(o(x)\) and \(o(x^{-1})\) are inverses
B3 No conclusion

Slips (-1)
S1 Arithmetic error

Attempts (2, 2 marks)
A1 \(x \circ x^{-1} = e\)
QUESTION 11

Part (a) 10 marks
Part (b) 20 (5, 5, 5, 5) marks
Part (c) 20 (5, 5, 5, 5) marks

Part (a) 10 marks

(a) Find the equation of the ellipse with foci \((\sqrt{7}, 0)\) and \((-\sqrt{7}, 0)\) and with eccentricity \(\frac{\sqrt{7}}{4}\).

\[
e = \frac{\sqrt{7}}{4} \quad \text{and} \quad ae = \sqrt{7} \quad \Rightarrow \quad a = 4.
\]

\[
b^2 = a^2\left(1 - e^2\right) = 16\left(1 - \frac{7}{16}\right) = 9.
\]

\[\therefore\text{ ellipse: } \frac{x^2}{16} + \frac{y^2}{9} = 1.
\]

Blunders (-3)
B1 Formula error

Slips (-1)
S1 Arithmetic error

Attempts (3 marks)
A1 \(ae = \sqrt{7}\) and stops
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(b) A transformation \( f \) is a similarity transformation if there exists a fixed number \( k \) such that \( |f(a)f(b)| = k|ab| \), for all \( a \) and \( b \).

Show that angle measure is invariant under a similarity transformation.

Part (b) \( f \) defined 5 marks Att 2

Use of \( k^2 \) 5 marks Att 2

Finish 5 marks Att 2

11 (b)

The triangle \( abc \) is mapped onto the triangle \( f(a)f(b)f(c) \) under a similarity transformation \( f \). \( \angle \theta \) is mapped onto \( \angle \phi \).

\[
\cos \angle \phi = \frac{|f(a)f(b)|^2 + |f(a)f(c)|^2 - |f(b)f(c)|^2}{2|f(a)f(b)||f(a)f(c)|} = \frac{k^2|ab|^2 + k^2|ac|^2 - k^2|bc|^2}{2k^2|ab||ac|}.
\]

\[
\therefore \cos \angle \phi = \frac{|ab|^2 + |ac|^2 - |bc|^2}{2|ab||ac|} = \cos \angle bac = \cos \theta.
\]

\[
\therefore |\angle \theta| = |\angle \phi|, \text{ since both are in the range } 0^\circ \text{ to } 180^\circ
\]

Hence angle measure is invariant under a similarly transformation.

Blunders (-3)

B1 Error in cosine formula
B2 Fails to identify \( |f(a)f(b)| = k|ab| \) or equivalent
B3 No conclusion or incorrect conclusion

Slips (-1)

S1 Arithmetic error

Attempts (2, 2, 2, 2 marks)

A1 \( \cos \phi \) with some substitution

Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(c) (i) Define the term conjugate diameters of an ellipse.

(ii) Prove that all parallelograms circumscribed to a given ellipse at the endpoints of conjugate diameters have the same area.
If $|pq|$ is a diameter of an ellipse $E$, then there is a second diameter $|rs|$, such that $|pq|$ bisects all chords of $E$ on lines parallel to $|rs|$ and vice versa. $|pq|$ and $|rs|$ are called conjugate diameters of the ellipse.

**Blunders (-3)**
B1 Parallel property not indicated

**Slips (-1)**
S1 Arithmetic error

**Attempts (2 marks)**
A1 Incomplete relevant diagram

---

**Part (c) (ii) $f$**

$\begin{align*}
&\begin{array}{c}
\text{pqrs} \\
\text{wzuv}
\end{array} \\
&f
\end{align*}$

$\begin{align*}
&\begin{array}{c}
\text{pqrs} \\
\text{wzuv}
\end{array} \\
&f^{-1}
\end{align*}$

Finish

$\begin{align*}
&\begin{array}{c}
\text{pqrs} \\
\text{wzuv}
\end{array} \\
&\text{Area} = \text{4 sq units}
\end{align*}$

$\begin{align*}
&\begin{array}{c}
\text{pqrs} \\
\text{wzuv}
\end{array} \\
&\text{Area} = \text{4 det}(f)$

$\begin{align*}
&\begin{array}{c}
\text{pqrs} \\
\text{wzuv}
\end{array} \\
&\text{area} = \text{4 det}(f)
\end{align*}$

But $\det(f)$ is constant $\Rightarrow$ area $p'q'r's'$ is constant.

$\therefore$ Areas of all parallelograms at end points of conjugate diameters of an ellipse are equal.

**Blunders (-3)**
B1 Fails to identify conjugate diameters of circle are perpendicular
B2 Fails to identify $\det(f)$ is constant and or area $pqrs$ is constant

**Slips (-1)**
S1 Arithmetic error

**Attempts (2, 2, 2 marks)**
A1 Some relevant mapping
MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAELILGE

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d’iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónaísin a shhlánú síos.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónaísin i gcás gach páipéir ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. 198 marc × 5% = 9·9 ⇒ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, riomhtar an bónas de réir na foirmle [300 – bunmharc] × 15%, agus an marc bónaísin a shhlánú síos. In ionad an ríomhaireachta sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

<table>
<thead>
<tr>
<th>Bunmharc</th>
<th>Marc Bónais</th>
</tr>
</thead>
<tbody>
<tr>
<td>226</td>
<td>11</td>
</tr>
<tr>
<td>227 – 233</td>
<td>10</td>
</tr>
<tr>
<td>234 – 240</td>
<td>9</td>
</tr>
<tr>
<td>241 – 246</td>
<td>8</td>
</tr>
<tr>
<td>247 – 253</td>
<td>7</td>
</tr>
<tr>
<td>254 – 260</td>
<td>6</td>
</tr>
<tr>
<td>261 – 266</td>
<td>5</td>
</tr>
<tr>
<td>267 – 273</td>
<td>4</td>
</tr>
<tr>
<td>274 – 280</td>
<td>3</td>
</tr>
<tr>
<td>281 – 286</td>
<td>2</td>
</tr>
<tr>
<td>287 – 293</td>
<td>1</td>
</tr>
<tr>
<td>294 – 300</td>
<td>0</td>
</tr>
</tbody>
</table>