Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2007

MATHEMATICS — HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 7 JUNE – MORNING, 9:30 to 12:00

Attempt SIX QUESTIONS (50 marks each).

WARNING: Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.
1. (a) Simplify \( \frac{x^2 - xy}{x^2 - y^2} \).

(b) Let \( f(x) = x^2 + (k+1)x - k - 2 \), where \( k \) is a constant.

(i) Find the value of \( k \) for which \( f(x) = 0 \) has equal roots.

(ii) Find, in terms of \( k \), the roots of \( f(x) = 0 \).

(iii) Find the range of values of \( k \) for which both roots are positive.

(c) \( x + p \) is a factor of both \( ax^2 + b \) and \( ax^2 + bx - ac \).

(i) Show that \( p^2 = \frac{-b}{a} \) and that \( p = \frac{-b - ac}{b} \).

(ii) Hence show that \( p^2 + p^3 = c \).

2. (a) Solve the simultaneous equations

\[
\begin{align*}
x + y + z &= 2 \\
2x + y + z &= 3 \\
x - 2y + 2z &= 15.
\end{align*}
\]

(b) \( \alpha \) and \( \beta \) are the roots of the equation \( x^2 - 4x + 6 = 0 \).

(i) Find the value of \( \frac{1}{\alpha} + \frac{1}{\beta} \).

(ii) Find the quadratic equation whose roots are \( \frac{1}{\alpha} \) and \( \frac{1}{\beta} \).

(c) (i) Prove that \( x + \frac{9}{x + 2} \geq 4 \), where \( x + 2 > 0 \).

(ii) Prove that \( x + \frac{9}{x + a} \geq 6 - a \), where \( x + a > 0 \).
3. \( \text{(a)} \) Let \( A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{4} \end{pmatrix} \). Find \( A^2 - 2A \).

\( \text{(b)} \) Let \( z = -1 + i \), where \( i^2 = -1 \).

(i) Use De Moivre’s theorem to evaluate \( z^5 \) and \( z^9 \).

(ii) Show that \( z^5 + z^9 = 12z \).

(c) \( \text{(i)} \) Find the two complex numbers \( a + bi \) for which \( (a + bi)^2 = 15 + 8i \).

(ii) Solve the equation \( iz^2 + (2 - 3i)z + (-5 + 5i) = 0 \).

4. \( \text{(a)} \) Show that \( \binom{n}{1} + \binom{n}{2} = \binom{n+1}{2} \) for all natural numbers \( n \geq 2 \).

\( \text{(b)} \) \( u_1 = 5 \) and \( u_{n+1} = \frac{n}{n+1}u_n \) for all \( n \geq 1, n \in \mathbb{N} \).

(i) Write down the value of each of \( u_2, u_3, \) and \( u_4 \).

(ii) Hence, by inspection, write an expression for \( u_n \) in terms of \( n \).

(iii) Use induction to justify your answer for part (ii).

\( \text{(c)} \) The sum of the first \( n \) terms of a series is given by \( S_n = n^2 \log_e 3 \).

(i) Find the \( n^{th} \) term and prove that the series is arithmetic.

(ii) How many of the terms of the series are less than \( 12 \log_e 27 \)?
5. (a) Plot, on the number line, the values of \( x \) that satisfy the inequality \( |x + 1| \leq 2 \), where \( x \in \mathbb{Z} \).

(b) In the expansion of \( \left( 2x - \frac{1}{x^2} \right)^9 \),

(i) find the general term

(ii) find the value of the term independent of \( x \).

(c) The \( n^{\text{th}} \) term of a series is given by \( nx^n \), where \(|x| < 1\).

(i) Find an expression for \( S_n \), the sum of the first \( n \) terms of the series.

(ii) Hence, find the sum to infinity of the series.

6. (a) Differentiate \( \frac{x^2 - 1}{x^2 + 1} \) with respect to \( x \).

(b) (i) Differentiate \( \frac{1}{x} \) with respect to \( x \) from first principles.

(ii) Find the equation of the tangent to \( y = \frac{1}{x} \) at the point \( (2, \frac{1}{2}) \).

(c) Let \( f(x) = \tan^{-1} \frac{x}{2} \) and \( g(x) = \tan^{-1} \frac{2}{x} \), for \( x > 0 \).

(i) Find \( f''(x) \) and \( g'(x) \).

(ii) Hence, show that \( f(x) + g(x) \) is constant.

(iii) Find the value of \( f(x) + g(x) \).
7. (a) Taking 1 as the first approximation of a root of \( x^3 + 2x - 4 = 0 \), use the Newton-Raphson method to calculate the second approximation of this root.

(b) (i) Find the equation of the tangent to the curve \( 3x^2 + y^2 = 28 \) at the point \((2, -4)\).

(ii) \( x = e^t \cos t \) and \( y = e^t \sin t \). Show that \( \frac{dy}{dx} = \frac{x + y}{x - y} \).

(c) \( f(x) = \log_e 3x - 3x \), where \( x > 0 \).

(i) Show that \((\frac{1}{3}, -1)\) is a local maximum point of \( f(x) \).

(ii) Deduce that the graph of \( f(x) \) does not intersect the \( x \)-axis.

8. (a) Find (i) \( \int x^3 \, dx \) (ii) \( \int \frac{1}{x^3} \, dx \).

(b) (i) Evaluate \( \int_0^4 \frac{1}{x} \sqrt{x^2 + 9} \, dx \).

(ii) \( f \) is a function such that \( f'(x) = 6 - \sin x \) and \( f\left(\frac{\pi}{3}\right) = 2\pi \).

Find \( f(x) \).

(c) The line \( 2x - y - 10 = 0 \) is a tangent to the curve \( y = x^2 - 9 \), as shown.

The shaded region is bounded by the line, the curve and the \( x \)-axis.

Calculate the area of this region.