LEAVING CERTIFICATE 2008

MARKING SCHEME

MATHEMATICS

HIGHER LEVEL
## Contents

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1. Penalties of three types are applied to candidates’ work as follows:
   - **Blunders** - mathematical errors/omissions (-3)
   - **Slips** - numerical errors (-1)
   - **Misreadings** (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any correct, relevant step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.
**QUESTION 1**

<table>
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<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
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<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
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<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
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<table>
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<th>Part (a)</th>
<th>10 (5, 5) marks</th>
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<tbody>
<tr>
<td>1. (a)</td>
<td>Simplify fully</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{x^2 + 4}{x^2 - 4} - \frac{x}{x + 2}$</td>
<td></td>
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</table>

**Correct Numerator** | 5 marks | Att 2 |
<table>
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<tbody>
<tr>
<td><strong>Finish</strong></td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
</tbody>
</table>

1 (a)

\[
\begin{align*}
\frac{x^2 + 4}{x^2 - 4} - \frac{x}{x + 2} &= \frac{x^2 + 4}{(x - 2)(x + 2)} - \frac{x}{x + 2} \\
&= \frac{(x^2 + 4) - x(x - 2)}{(x - 2)(x + 2)} \\
&= \frac{x^2 + 4 - x^2 + 2x}{(x - 2)(x + 2)} \\
&= \frac{2x + 4}{(x - 2)(x + 2)} \\
&= \frac{2(x + 2)}{(x - 2)(x + 2)} = \frac{2}{x - 2}
\end{align*}
\]

**Blunders (-3)**

- B1 Factors once only
- B2 Indices
- B3 Incorrect cancellation
1. (b) Given that one of the roots is an integer, solve the equation
\[ 6x^3 - 29x^2 + 36x - 9 = 0. \]

<table>
<thead>
<tr>
<th>Part (b)</th>
<th>20 (5, 5, 5) marks</th>
<th>Att (2, 2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. (b)</strong></td>
<td></td>
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</tr>
<tr>
<td>(b)</td>
<td></td>
<td></td>
</tr>
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</table>

Getting \((x - 3)\) as factor: 5 marks
Division: 5 marks
Remaining two factors: 5 marks
Roots: 5 marks

\[
f(x) = 6x^3 - 29x^2 + 36x - 9
\]

\[
f(1) = 6 - 29 + 36 - 9 \neq 0
\]

\[
f(2) = 48 - 116 + 72 - 9 \neq 0
\]

\[
f(3) = 162 - 261 + 108 - 9 = 270 - 270 = 0.
\]

\[ \therefore x = 3 \Rightarrow (x - 3) \text{ is a factor.} \]

\[
(x - 3)(6x^2 + ax + 3) = 6x^3 - 29x^2 + 36x - 9.
\]

\[ \therefore a + 18 = -29 \Rightarrow a = -11. \]

\[ \therefore 6x^2 - 11x + 3 = 0 \Rightarrow (3x - 1)(2x - 3) = 0. \]

\[ \therefore 3x - 1 = 0 \text{ or } 2x - 3 = 0 \Rightarrow x = \frac{1}{3} \text{ or } x = \frac{3}{2}. \]

Roots are \(3, \frac{1}{3}, \frac{3}{2}.\)

OR
Getting \((x - 3)\) as factor 5 marks  
Division 5 marks  
Remaining two factors 5 marks  
Roots 5 marks  

1. \(b\)

\[
 f(x) = 6x^3 - 29x^2 + 36x - 9 \\
 f(1) = 6 - 29 + 36 - 9 \neq 0 \\
 f(-1) \neq 0 \\
 f(3) = 6(27) - 29(9) + 36(3) - 9 \\
 = 162 - 261 + 108 - 9 \\
 = 270 - 270 \\
 f(3) = 0 \quad \Rightarrow (x - 3) \text{ is a factor}
\]

\[
 x - 3 \overline{6x^3 - 29x^2 + 36x - 9} \\
 \underline{6x^3 - 18x^2} \\
 -11x^2 + 36x \\
 \underline{-11x^2 + 33x} \\
 3x - 9 \\
 \underline{3x - 9} \\
 f(x) = (x - 3)(6x^2 - 11x + 3) \\
 = (x - 3)(3x - 1)(2x - 3) \\
 f(x) = 0 \Rightarrow (x - 3)(3x - 1)(2x - 3) = 0 \\
 \Rightarrow x = 3, \frac{1}{3}, \frac{3}{2}
\]

Blunders (-3)
B1 Test for root
B2 Deduction of factor from root or no deduction
B3 Indices
B4 Root formula (once only)
B5 Deduction of root from factor or no deduction
B6 Not like to like when equating coefficients

Slips (-1)
S1 Numerical
S2 Not changing sign when subtracting in division

Worthless
W1 \(x(6x^3 - 29x + 36) = 9\), with or without further work

NOTE If there is a remainder after division, or incomplete division, candidates can only get Att at most for remaining factors and roots.
Two of the roots of the equation \( ax^3 + bx^2 + cx + d = 0 \) are \( p \) and \( -p \).

Show that \( bc = ad \).

\[
(x^2 - p^2) \text{ a factor} \quad 5 \text{ marks} \\
\text{Divison} \quad 5 \text{ marks} \\
\text{Remainder} = 0 \quad 5 \text{ marks} \\
\text{Finish} \quad 5 \text{ marks}
\]

\[
1. (c) \\
x = p \quad \text{and} \quad x = -p \quad \Rightarrow \quad (x - p)(x + p) = x^2 - p^2 \text{ is a factor.} \\
ax^3 + bx^2 + cx + d = (x^2 - p^2) \left( ax - \frac{d}{p^2} \right). \\
\therefore \quad b = -\frac{d}{p^2} \quad \text{and} \quad c = -ap^2. \\
p^2 = -\frac{c}{a} \quad \Rightarrow \quad b = \frac{ad}{c} \quad \Rightarrow \quad bc = ad.
\]

OR
1 (c) 

\[ p \text{ and } (-p) \text{ are roots } \Rightarrow (x - p) \text{ and } (x + p) \text{ are factors} \]
\[ \Rightarrow (x^2 - p^2) \text{ is a factor} \]

\[
\frac{ax + b}{x^2 - p^2} \frac{ax^3 + bx^2 + cx + d}{ax^3 - ap^2x} = \frac{bx^2 + (c + ap^2)x}{bx^2 - bp^2} = \frac{(c + ap^2)x + bp^2 + d}{(c + ap^2)x + bp^2 + d}
\]

Since \((x^2 - p^2)\) is factor, remainder = 0

\[ (c + ap^2)x + (bp^2 + d) = (0)x + (0) \]
\[ \Rightarrow (i): c + ap^2 = 0 \]
\[ p^2 = -\frac{c}{a} \]

(ii) \[ bp^2 + d = 0 \]
\[ p^2 = -\frac{d}{b} \]

From (i) and (ii): \[-\frac{c}{a} = -\frac{d}{b} \]
\[ cb = ad \]

OR
1 (c) Since \( p \) is a root of \( ax^3 + bx^2 + cx + d = 0 \)
then \( a(p)^3 + b(p)^2 + c(p) + d = 0 \)
\[ ap^3 + bp^2 + cp + d = 0 \] ...............(i)

Similarly \((- p)\) is a root
\[ a(-p)^3 + b(-p)^2 + c(-p) + d = 0 \]
\[ -ap^3 + bp^2 - cp + d = 0 \] ...............(ii)

Adding (i) and (ii):
\[ 2bp^2 + 2d = 0 \]
\[ bp^2 = -d \]
\[ p^2 = -\frac{d}{b} \] ...............(iii)

Subtracting (i) and (ii):
\[ 2ap^3 + 2cp = 0 \]
\[ ap^3 = -c \]
\[ p^2 = -\frac{c}{a} \] ...............(iv)

From (iii) and (iv):
\[ -\frac{d}{b} = -\frac{c}{a} \]
\[ ad = bc \]

Blunders (-3)
B1 Indices
B2 Factor not \((x^2 - p^2)\) once only
B3 Not like to like when equating coefficients

Slips (-1)
S1 Not changing sign when subtracting in division

Attempts
A1 Any effort at division
A2 Other factor not linear – cannot now get any more marks
QUESTION 2

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</tr>
</tbody>
</table>

2. (a) 

(a) Express \( x^2 + 10x + 32 \) in the form \( (x + a)^2 + b \).

\[
\begin{align*}
x^2 + 10x + 32 &= x^2 + 10x + 25 + 7 = (x + 5)^2 + 7.
\end{align*}
\]

* Accept solutions based on two values of \( x \)

**OR**

2. (a) 

Equating Coefficients 5 marks Att 2
Solving equations 5 marks Att 2

\[
\begin{align*}
x^2 + 10x + 32 &= (x + a)^2 + b \\
x^2 + (10)x + 32 &= x^2 + (2a)x + (a^2 + b)
\end{align*}
\]

(i) \( 10 = 2a \) 
\( 5 = a \)

(ii) \( a^2 + b = 32 \) 
\( 25 + b = 32 \)

\( b = 7 \)

Blunders (-3)
B1 Indices
B2 Expansion of \((x + a)^2\) once only
B3 Completing square
B4 Not like to like when equating coefficients
B5 No ‘a’ or no deduction ‘a’
B6 No ‘b’ or no deduction ‘b’

Slips (-1)
S1 Numerical
Part (b)

20 (5, 5, 5, 5) marks

Att (2, 2, 2, 2)

2. (b) \( \alpha \) and \( \beta \) are the roots of the equation \( x^2 - 7x + 1 = 0 \).

(i) Find the value of \( \alpha^2 + \beta^2 \).

(ii) Find the value of \( \frac{1}{\alpha^3} + \frac{1}{\beta^3} \).

\[(i) \text{ Values: } \alpha + \beta \text{ & } \alpha \beta \text{, or solve quad. } 5 \text{ marks} \quad \text{Att 2}
\]

\[\alpha^2 + \beta^2 \quad 5 \text{ marks} \quad \text{Att 2} \]

\[(ii) \text{ Factors } 5 \text{ marks} \quad \text{Att 2}
\]

\[\text{Value} \quad 5 \text{ marks} \quad \text{Att 2} \]

\[2. (b) (i) \]

\[\alpha + \beta = \frac{-b}{a} = 7 \text{ and } \alpha \beta = \frac{c}{a} = 1. \]

\[(\alpha + \beta)^2 = 49 \Rightarrow \alpha^2 + \beta^2 + 2\alpha \beta = 49. \]

\[\therefore \alpha^2 + \beta^2 = 47. \]

\[2. (b) (ii) \]

\[\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{\alpha^3 \beta^3} = \frac{(\alpha + \beta)(\alpha^2 - \alpha \beta + \beta^2)}{1} \]

\[= 7(47 - 1) = 322. \]

**Blunders (-3)**

B1 Indices
B2 Incorrect sum
B3 Incorrect product
B4 Statement incorrect
B5 Factors

**Slips (-1)**

S1 Numerical
2. (c) Show that if \( a \) and \( b \) are non-zero real numbers, then the value of \( \frac{a}{b} + \frac{b}{a} \) can never lie between \(-2\) and \(2\).

Hint: consider the case where \( a \) and \( b \) have the same sign separately from the case where \( a \) and \( b \) have opposite sign.

<table>
<thead>
<tr>
<th>Case 1: ( a ) and ( b ) have same sign.</th>
</tr>
</thead>
<tbody>
<tr>
<td>In this case, ( \frac{a}{b} + \frac{b}{a} &gt; 0 ), so we need to show that ( \frac{a}{b} + \frac{b}{a} &gt; 2 ).</td>
</tr>
<tr>
<td>( \frac{a}{b} + \frac{b}{a} &gt; 2 )</td>
</tr>
<tr>
<td>( \frac{a}{b} + \frac{b}{a} &gt; 2 )</td>
</tr>
<tr>
<td>( \Leftrightarrow a^2 + b^2 &gt; 2ab ) , (since ( ab &gt; 0 ))</td>
</tr>
<tr>
<td>( \Leftrightarrow a^2 - 2ab + b^2 &gt; 0 )</td>
</tr>
<tr>
<td>( \Leftrightarrow (a - b)^2 &gt; 0 ) True.</td>
</tr>
</tbody>
</table>

Case 2: \( a \) and \( b \) have opposite sign.

In this case, \( \frac{a}{b} + \frac{b}{a} < 0 \), so we need to show that \( \frac{a}{b} + \frac{b}{a} < -2 \).

\( \frac{a}{b} + \frac{b}{a} < -2 \)

\( \Leftrightarrow a^2 + b^2 > -2ab \) , (since \( ab < 0 \))

\( \Leftrightarrow a^2 + 2ab + b^2 > 0 \)

\( \Leftrightarrow (a + b)^2 > 0 \) True.

Or
\[ x + \frac{1}{x} = k \quad \text{5 marks} \quad \text{Att 2} \]

**Quadratic** \quad 5 marks \quad Att 2

\[ b^2 - 4ac \quad \text{5 marks} \quad \text{Att 2} \]

**Deduction** \quad 5 marks \quad Att 2

\[ 2.\text{(c)} \]

Let \( \frac{a}{b} = x \). Then must show \( x + \frac{1}{x} \) is never \( \in [-2,2] \)

Let \( x + \frac{1}{x} = k \). So, need to show \( |k| > 2 \)

\[
\begin{align*}
  x + \frac{1}{x} &= k \\
  x^2 + 1 &= kx \\
  x^2 - kx + 1 &= 0
\end{align*}
\]

For real \( x \), \( b^2 - 4ac > 0 \)

i.e. \( k^2 - 4 > 0 \)

\( k^2 > 4 \)

i.e., \( |k| > 2 \)

---

**OR**

**Mod Value** \quad 5 marks \quad Att 2

**Squaring** \quad 5 marks \quad Att 2

\[
\left( \frac{a}{b} - \frac{b}{a} \right)^2
\]

5 marks \quad Att 2

**Deduction** \quad 5 marks \quad Att 2

\[ 2.\text{(c)} \]

Need to show. \( \left| \frac{a}{b} + \frac{b}{a} \right| > 2 \)

Proof: \( \iff \left( \frac{a}{b} + \frac{b}{a} \right)^2 > 4 \)

\[
\begin{align*}
  \iff \frac{a^2}{b^2} + \frac{b^2}{a^2} + 2 &> 4 \\
  \iff \frac{a^2}{b^2} + \frac{b^2}{a^2} - 2 &> 0 \\
  \iff \left( \frac{a}{b} - \frac{b}{a} \right)^2 &> 0 \quad \text{True}
\end{align*}
\]

---

**Blunders (-3)**

B1  Inequality sign

B2  Factors

B3  Incorrect deduction or no deduction
### QUESTION 3

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<td>Att (2, 2, 2, 2)</td>
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#### Part (a) 10 (5, 5) marks Att (2, 2)

3. (a) Let \( A \) be the matrix \[
\begin{pmatrix}
3 & 5 \\
1 & 2
\end{pmatrix}
\]
Find the matrix \( B \), such that \( AB = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} \).

\[
A = \begin{pmatrix}
3 & 5 \\
1 & 2
\end{pmatrix} \Rightarrow A^{-1} = \frac{1}{6-5} \begin{pmatrix}
2 & -5 \\
-1 & 3
\end{pmatrix} = \begin{pmatrix}
2 & -5 \\
-1 & 3
\end{pmatrix}
\]

\[
B = A^{-1} \begin{pmatrix}
4 & 6 \\
3 & 2
\end{pmatrix} = \begin{pmatrix}
2 & -5 \\
-1 & 3
\end{pmatrix} \begin{pmatrix}
4 & 6 \\
3 & 2
\end{pmatrix} = \begin{pmatrix}
-7 & 2 \\
5 & 0
\end{pmatrix}
\]

#### OR

Four equations 5 marks Att 2
Four values 5 marks Att 2

3 (a) Let \( B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \) Then \( AB = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} \)

(i): \( 3p + 5r = 4 \) \hspace{1cm} (ii): \( 3q + 5s = 6 \)

(iii): \( p + 2r = 3 \) \hspace{1cm} (iv): \( q + 2s = 2 \)

(i) and (iii): \( 3p + 5r = 4 \Rightarrow 3p + 5r = 4 \\
p + 2r = 3 \Rightarrow 3p + 6r = 9 \\
-5r = 5 \Rightarrow r = 5 \Rightarrow p = -7 \)

(ii) and (iv): \( 3q + 5s = 6 \Rightarrow 3q + 5s = 6 \\
q + 2s = 2 \Rightarrow 3q + 6s = 6 \\
-s = 0 \Rightarrow s = 0 \Rightarrow q = 2 \)

\[
B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ 5 & 0 \end{pmatrix}
\]

**Blunders (-3)**
B1 Formula for inverse
B2 Matrix multiplication

**Slips (-1)**
S1 Each incorrect element
S2 Numerical
3. (b) (i) Let \( z = \frac{5}{2 + i} - 1 \), where \( i^2 = -1 \).

Express \( z \) in the form \( a + bi \) and plot it on an Argand diagram.

(ii) Use De Moivre’s theorem to evaluate \( z^6 \).

(i) z when multiplied by conjugate 5 marks  
Plot 5 marks  
(ii) z in polar form 5 marks  
Value 5 marks

3 (b) (i)

\[
z = \frac{5}{2 + i} - 1 = \frac{5 - (2 + i)}{2 + i} = \frac{3 - i}{2 + i} = \frac{3 - i}{2 + i} \cdot \frac{2 - i}{2 + i} = \frac{6 - 5i + i^2}{4 - i^2} = \frac{5 - 5i}{5} \\
\Rightarrow z = 1 - i
\]

3 (b) (ii)

\[
z = r(\cos \theta + i \sin \theta) \\
z = 2^{\frac{1}{2}} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \\
z^6 = \left[ 2^{\frac{1}{2}} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \right]^6 \\
= (2)^6 \left[ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right] \\
= 8 \left[ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right] \\
= 8i
\]

\[
\theta = \frac{3\pi}{4} \\
|z| = r = \sqrt{1 + (-1)^2} \\
r = \sqrt{2} = 2^{\frac{1}{2}}
\]

Blunders (-3)
B1 Indices  
B2 i  
B3 \((2 + i)(2 - i) \neq 5\)  
B4 Argument  
B5 Modulus  
B6 Trig Definition  
B7 Statement De Moivre once only  
B8 Application De Moivre  
B9 No plot z or incorrect plot z

Slips (-1)
S1 Trig value  

Worthless
W1 Not De Moivre
### 3 (c) Prove, by induction, that

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad \text{for } n \in \mathbb{N}.\]

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<tbody>
<tr>
<td><strong>P(1) or P(0)</strong></td>
<td></td>
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<tr>
<td><strong>P(k)</strong></td>
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<tr>
<td><strong>P(k + 1)</strong></td>
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<tr>
<td><strong>Proof</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Proof:

Prove: \[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \quad [n \in \mathbb{N}]\]

Test \(n = 0\): \[(\cos \theta + i \sin \theta)^0 = \cos 0\theta + i \sin 0\theta\]

\[1 = \cos 0 + i \sin 0\]

\[1 = 1\]

True for \(n = 0\)

Assume true for \(n = k\): \[(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta\]

To prove: \[(\cos \theta + i \sin \theta)^{k+1} = \cos (k+1)\theta + i \sin (k+1)\theta\]

\[
(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) \quad \text{(by ind. hyp.)} \\
= \cos(k+1)\theta + i \sin(k+1)\theta
\]

So, \{true for \(n = k\) \implies true for \(n = k + 1\)\}

\[\therefore \text{True for all } n \in \mathbb{N}.\]

---

**Blunders (-3)**

- B1 Indices
- B2 Trig Formula
- B3 \(i\)
- B4 Statement De Moivre

---

*NOTE: Accept \(n = 0\) or \(n = 1\) for first step*
**QUESTION 4**

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

4. (a) \(2 + \frac{2}{3} + \frac{2}{9} + \ldots\) is a geometric series.

Find the sum to infinity of the series.

<table>
<thead>
<tr>
<th>Correct substitution into formula</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
</tbody>
</table>

### 4 (a)

\[
S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = 3.
\]

**Blunders (-3)**

- B1: Formula sum to infinity
- B2: Indices
- B3: Incorrect ‘a’
- B4: Incorrect ‘r’

**Worthless**

- W1: Uses A.P.
4 (b)  Given that \( u_n = 2 \left( -\frac{1}{2} \right)^n - 2 \) for all \( n \in \mathbb{N} \),

(i) write down \( u_{n+1} \) and \( u_{n+2} \)

(ii) show that \( 2u_{n+2} - u_{n+1} - u_n = 0. \)

<table>
<thead>
<tr>
<th>(i) Write down</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ii) Terms simplified</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
<tr>
<td>Correct substitution</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
<tr>
<td>Finish</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
</tbody>
</table>

4 (b) (i) \[
u_{n+1} = 2 \left( -\frac{1}{2} \right)^{n+1} - 2, \quad u_{n+2} = 2 \left( -\frac{1}{2} \right)^{n+2} - 2.\]

(ii) \[
2u_{n+2} - u_{n+1} - u_n = 4 \left( -\frac{1}{2} \right)^{n+2} - 4 - 2 \left( -\frac{1}{2} \right)^{n+1} + 2 \left( -\frac{1}{2} \right)^n + 2.
\]

\[
= 4 \left( 1 - \frac{1}{2} \right)^n - 2 \left( -\frac{1}{2} \right)^n - 2 \left( -\frac{1}{2} \right)^n
\]

\[
= \left( -\frac{1}{2} \right)^n + \left( -\frac{1}{2} \right)^n - 2 \left( -\frac{1}{2} \right)^n = 0.
\]

**Blunders (-3)**

B1  Indices

**Attempts**

A1  Must do some correct relevant work with indices

NOTE: Simplification and substitution can be in any order
4 (c) (i) Write down an expression in \( n \) for the sum \( 1 + 2 + 3 + \ldots + n \)
and an expression in \( n \) for the sum \( 1^2 + 2^2 + 3^2 + \ldots + n^2 \).

(ii) Find, in terms of \( n \), the sum \( \sum_{r=1}^{n} (6r^2 + 2r + 5 + 2^r) \).

(i) Formulae 5 marks
(ii) 1st two terms 5 marks
5n 5 marks
G.P. 5 marks

4 (c) (i)

\[
1 + 2 + 3 + \ldots + n = \sum_{1}^{n} n = \frac{n(n+1)}{2}.
\]

\[
1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

(ii)

\[
\sum_{r=1}^{n} (6r^2 + 2r + 5 + 2^r) = 6\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 5 + \sum_{r=1}^{n} 2^r
\]

\[
= n(n+1)(2n+1) + n(n+1) + 5n + \frac{2(2^n - 1)}{2 - 1}.
\]

\[
= n(n+1)(2n+1) + n(n+1) + 5n + 2(2^n - 1)
\]

Blunders (-3)
B1 Indices
B2 Incorrect \( \sum n \)
B3 Incorrect \( \sum n^2 \)
B4 5n term
B5 Formula G.S
B6 Incorrect 'a'
B7 Incorrect \( r \)

Slips (-1)
S1 Numerical
5. (a) Find the range of values of \( x \) that satisfy the inequality 
\[ x^2 - 3x - 10 \leq 0. \]

Factors 5 marks  
Range 5 marks  

\[ 5 \text{ (a)} \]
\[ x^2 - 3x - 10 \leq 0 \Rightarrow (x - 5)(x + 2) \leq 0. \]

Graph: \( x \rightarrow (x - 5)(x + 2) \)
\[ f(x) \leq 0 \text{ when } -2 \leq x \leq 5 \]

OR

Factors 5 marks  
Range 5 marks  

\[ 5 \text{ (a)} \]
\[ x^2 - 3x - 10 \leq 0 \]
\[ (x - 5)(x + 2) \leq 0 \]

Either: I: \( x - 5 \geq 0 \) and \( x + 2 \leq 0 \)
\[ x \geq 5 \text{ and } x \leq -2 \]
Not Possible

or: II: \( x - 5 \leq 0 \) and \( x + 2 \geq 0 \)
\[ x \leq 5 \text{ and } x \geq -2 \]

\[ \therefore \text{ answer is } -2 \leq x \leq 5 \]

Blunders (-3)
B1 Factors  
B2 Root from factor  
B3 Upper Limit  
B4 Lower Limit  
B5 Inequality sign  
B6 Root formula, once only  
B7 Incorrect range  
B8 Answer not stated

Slips (-1)
S1 Numerical

Attempts
A1 One inequality sign  
A2 Inequality signs ignored
### Part (b) 20 (5, 5, 5, 5) marks  
Att (2, 2, 2, 2)

<table>
<thead>
<tr>
<th>5 (b)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>Solve the equation ( 2^x = 8^{2x+9} ).</td>
</tr>
<tr>
<td>(ii)</td>
<td>Solve the equation ( \log_e(2x+3) + \log_e(x-2) = 2 \log_e(x+4) ).</td>
</tr>
</tbody>
</table>

#### (b)(i) Quadratic

**Solve**

<table>
<thead>
<tr>
<th>5 (b) (i)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^x = 8^{2x+9} )</td>
<td>( \Rightarrow 2^x = 2^{6x+27} ).</td>
</tr>
<tr>
<td>( \therefore x^2 - 6x - 27 = 0 )</td>
<td>( \Rightarrow (x - 9)(x + 3) = 0 ).</td>
</tr>
<tr>
<td>( \therefore x = 9 \text{ or } x = -3. )</td>
<td></td>
</tr>
</tbody>
</table>

**Blunders (-3)**

- B1 Indices
- B2 Factors
- B3 Root formula, once only
- B4 Deduction root from factor

#### (b)(ii) Correct working with logs

**Correct value \( x \)**

<table>
<thead>
<tr>
<th>5 (b) (ii)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_e(2x+3) + \log_e(x-2) = 2 \log_e(x+4) )</td>
<td>( \log_e(2x+3)(x-2) = \log_e(x+4)^2 ).</td>
</tr>
<tr>
<td>( \therefore 2x^2 - x - 6 = x^2 + 8x + 16 )</td>
<td>( \therefore x - 9x - 22 = 0 \Rightarrow (x-11)(x+2) = 0 ).</td>
</tr>
<tr>
<td>( \therefore x = 11, x = -2. )</td>
<td></td>
</tr>
</tbody>
</table>

Test: \( x = 11 \)  
L.H.S.: \( \ln(25) + \ln(9) = \ln 225 \)  
R.H.S.: \( 2 \ln(15) = \ln 225 \)

Test: \( x = -2 \)  
L.H.S.: \( \ln(-1) + \ln(-4) \), which do not exist

\( \therefore \) the only solution is \( x = 11. \)

**Blunders (-3)**

- B1 Logs
- B2 Indices
- B3 Factors
- B4 Root Formula
- B5 Deduction root from factor or no deduction
- B6 Excess value

**Worthless**

- W1 Drops ‘Log’
Part (c) 20 (5, 5, 5) marks  
Att (2, 2, 2)

5 (c) Show that there are no natural numbers \(n\) and \(r\) for which \(\binom{n}{r-1} \binom{n}{r} \text{ and } \binom{n}{r+1}\) are consecutive terms in a geometric sequence.

| Definition of G.S. | 5 marks | Att 2 |
| Factorial values inserted | 5 marks | Att 2 |
| Simplified fractions | 5 marks | Att 2 |
| Not natural no | 5 marks | Att 2 |

5 (c)

If a geometric sequence, then \(\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{\binom{n}{r+1}}{\binom{n}{r}}\).

\[
\quad \frac{n!}{r!(n-r)!} = \frac{\binom{n}{r+1}}{\binom{n}{r}} = \frac{n!}{(r+1)!(n-r-1)!} \frac{n!}{r!(n-r)!}.
\]

\[
\therefore \quad \frac{n-r+1}{r} = \frac{n-r}{r+1} \Rightarrow (n-r+1)(r+1) = r(n-r).
\]

\[
\therefore \quad nr + n - r^2 - r + r + 1 = nr - r^2 \Rightarrow n = -1, \text{ which is not a natural number.}
\]

Blunders (-3)

B1 Definition of G.S.
B2 Incorrect \(\binom{n}{r}\)
B3 Incorrect \(\binom{n}{r-1}\)
B4 Incorrect \(\binom{n}{r+1}\)
B5 Factorial
B6 Indices
B7 Cross multiplication
B8 Incorrect deduction or no deduction
### QUESTION 6

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 10) marks</td>
<td>Att (2, 2, 3)</td>
</tr>
</tbody>
</table>

### Part (a) 10 marks

6. (a) Differentiate $\sqrt{x^3}$ with respect to $x$.

\[
\begin{align*}
6 \text{ (a)} & \\
& f(x) = x^{\frac{3}{2}} \Rightarrow f'(x) = \frac{3}{2}x^{\frac{1}{2}}.
\end{align*}
\]

**Blunders** (-3)

- B1 Blunder indices
- B2 Blunder differentiation

### Part (b) 20 (5, 5, 5, 5) marks

6 (b) Let $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Show that \[
\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}.
\]

\[
\begin{align*}
\frac{dy}{dx} & = \frac{(e^x + e^{-x})(e^x - e^{-x}) - (e^x - e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2} \\
& = \frac{2e^x + 2e^{-2x} - e^{2x} + 2 - e^{-2x}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}.
\end{align*}
\]

OR
\[
y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \frac{e^{2x} - 1}{e^{2x} + 1}
\]

\[
\frac{dy}{dx} = \frac{(e^{2x} + 1)(2e^{2x}) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}
\]
\[
= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x} + 1)^2}
\]
\[
= \frac{4e^{2x}}{(e^{2x} + 1)^2}
\]
\[
= 4 \cdot \frac{1}{e^{-2x}} \cdot \frac{1}{(e^{2x} + 1)} \cdot \frac{1}{(e^{2x} + 1)}
\]
\[
= 4 \cdot \frac{1}{e^{-x}(e^{2x} + 1)} \cdot \frac{1}{e^{-x}(e^{2x} + 1)}
\]
\[
= 4 \left( \frac{1}{e^x + e^{-x}} \right) \left( \frac{1}{e^x + e^{-x}} \right)
\]
\[
= \frac{4}{(e^x + e^{-x})^2}
\]

**Blunders (-3)**

B1 Indices

B2 Differentiation

**Worthless**

W1 No differentiation

W2 Integration
The function $f(x) = 2x^3 + 3x^2 + bx + c$ has a local maximum at $x = -2$.

(i) Find the value of $b$.

(ii) Find the range of values of $c$ for which $f(x) = 0$ has three distinct real roots.

| (i) value $b$ | 5 marks | Att 2 |
| (ii) Local min at $x = 1$ | 5 marks | Att 2 |
| Range $c$ | 10 marks | Att 3 |

6 (c) (i)

\[ f(x) = 2x^3 + 3x^2 + bx + c \]
\[ f'(x) = 6x^2 + 6x + b \]

Local max at $x = -2$ \implies $f'(-2) = 0$

\[ 6(-2)^2 + 6(-2) + b = 0 \]
\[ 24 - 12 + b = 0 \]
\[ b = -12 \]

6 (c) (ii)

\[ f(x) = 2x^3 + 3x^2 - 12x + c \]
\[ f'(x) = 6x^2 + 6x - 12 = 0 \] for local max/min
\[ x^2 + x - 2 = 0 \]
\[ (x + 2)(x - 1) = 0 \]
\[ x = -2 \quad \text{or} \quad x = 1 \]

We were given that local max is at $x = -2$, so local min is at $x = 1$.

To get 3 distinct real roots, the curve must cut the $x$-axis 3 times.

Hence, we need the local max to be above the $x$-axis and the local min below it.

Local max: $x = -2$: $f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) = -16 + 12 + 24 + c = c + 20$

Above $x$-axis \implies $f(-2) > 0$ \implies $c + 20 > 0$ \implies $c > -20$

Local min at $x = 1$: $f(1) = 2(1)^3 + 3(1)^2 - 12(1) + c = c - 7$

Below $x$-axis \implies $f(1) < 0$ \implies $c - 7 < 0$ \implies $c < 7$

Thus, answer is $-20 < c < 7$

* Candidates need not explicitly state that local max and local min are on opposite sides of $x$-axis.

**Blunders (-3)**
B1 Differentiation
B2 $f'(x) \neq 0$
B3 Indices
B4 Factors
B5 Root formula once only
B6 Deducted root from factor or no deduction
B7 Inequality sign
B8 Incorrect range or no range

**Slips (-1)**
S1 Numerical

**Worthless**
W1 No Differentiation
W2 Integration
QUESTION 7

| Part (a) | 10 (5, 5) marks | Att (2, 2) |
| Part (b) | 20 (5, 5, 5, 5) marks | Att (2, 2, 2, 2) |
| Part (c) | 20 (15, 5) marks | Att (5, 2) |

7. (a) Differentiate $2x + \sin 2x$ with respect to $x$.

$f'(2x)$ 5 marks  Att 2
$f'(\sin 2x)$ 5 marks  Att 2

7 (a) $f(x) = 2x + \sin 2x \Rightarrow f'(x) = 2 + 2\cos 2x.$

Blunders (-3)
B1 Differentiation
B2 Trig formula

Attempts
A1 Error in chain rule

Worthless
W1 Integration
<table>
<thead>
<tr>
<th>Part (b)</th>
<th>20 (5, 5, 5, 5) marks</th>
<th>Att (2, 2, 2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 (b)</td>
<td>The equation of a curve is $5x^2 + 5y^2 + 6xy = 16$.</td>
<td></td>
</tr>
<tr>
<td>(i)</td>
<td>Find $\frac{dy}{dx}$ in terms of $x$ and $y$.</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>$(1,1)$ and $(2, -2)$ are two points on the curve. Show that the tangents at these points are perpendicular to each other.</td>
<td></td>
</tr>
</tbody>
</table>

(i) Differentiation 5 marks Att 2
Isolate $\frac{dy}{dx}$ 5 marks Att 2
(ii) 1st slope 5 marks Att 2
Show 5 marks Att 2

7 (b) (i)

\[ 5x^2 + 5y^2 + 6xy = 16. \]
\[ \therefore 10x + 10y \frac{dy}{dx} + 6x \frac{dy}{dx} + 6y = 0. \]
\[ \therefore \frac{dy}{dx} (10y + 6x) = -10x - 6y \Rightarrow \frac{dy}{dx} = \frac{-5x - 3y}{3x + 5y}. \]

7 (b) (ii)

\[ m_1 = \text{slope of tangent at } (1,1) = \frac{-5 - 3}{3 + 5} = -1. \]
\[ m_2 = \text{slope of tangent at } (2, -2) = \frac{-10 + 6}{6 - 10} = 1. \]
But $m_1, m_2 = -1$, \therefore tangents are perpendicular to each other.

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Incorrect value of $x$ or no value of $x$
B4 Incorrect value of $y$ or no value of $y$
B5 Omission of $m_1, m_2$ test

Slips (-1)
S1 Numerical

Attempts
A1 Error in differentiation formula
A2 $\frac{dy}{dx} = 10x + 10y \frac{dy}{dx} + 6x \frac{dy}{dx} + 6y$
And uses the three $\left( \frac{dy}{dx} \right)$ terms

Worthless
W1 No differentiation
W2 Integration
7 (c) Let \( y = \sin^{-1}\left( \frac{x}{\sqrt{1+x^2}} \right) \).

Find \( \frac{dy}{dx} \) and express it in the form \( \frac{a}{a+x^b} \), where \( a, b \in \mathbb{N} \).

\[ \tan y = x \quad \text{Differentiate} \]

\[ 5 \text{ marks} \quad \text{Att 2} \]

\[ 15 \text{ marks} \quad \text{Att 5} \]

\[ 7(c) \quad y = \sin^{-1}\left( \frac{x}{\sqrt{1+x^2}} \right) \Rightarrow \sin y = \frac{x}{\sqrt{1+x^2}} \]

\[ \tan y = \frac{x}{1} = x \]

\[ y = \tan^{-1} x \]

\[ \therefore \frac{dy}{dx} = \frac{1}{1+x^2} \]

\[ \text{OR} \]

\[ \text{Differentiation} \quad 15 \text{ marks} \quad \text{Att 5} \]

\[ \text{Other work} \quad 5 \text{ marks} \quad \text{Att 2} \]

\[ 7 (c) \]

\[ y = \sin^{-1} \frac{x}{\sqrt{1+x^2}} \]

\[ \sin y = \frac{x}{\sqrt{1+x^2}} = \frac{x}{(1+x^2)^{\frac{1}{2}}} \]

\[ \cos y \cdot \frac{dy}{dx} = \frac{(1+x^2)^{\frac{1}{2}}(1) - x\left(\frac{1}{2}(1+x^2)^{\frac{1}{2}} \cdot 2x\right)}{(1+x^2)} \]

\[ = \frac{(1+x^2)^{\frac{1}{2}} - \frac{x^2}{(1+x^2)^{\frac{1}{2}}}}{(1+x^2)^{\frac{1}{2}}} \]

\[ = \frac{1+x^2-x^2}{(1+x^2)^{\frac{1}{2}}} \]

\[ = \frac{1}{(1+x^2)^{\frac{1}{2}}} \]

\[ \cos y \cdot \frac{dy}{dx} = \frac{1}{(1+x^2)^{\frac{1}{2}}} \]

\[ \frac{dy}{dx} = \frac{1}{\cos y \cdot (1+x^2)^{\frac{1}{2}}} \]

\[ = \left(1+x^2\right)^{\frac{1}{2}} \cdot \frac{1}{(1+x^2)^{\frac{1}{2}}} \]

\[ \frac{dy}{dx} = \frac{1}{1+x^2} \]
\begin{align*}
7 \text{ (c)} & \quad y = \sin^{-1}\left(\frac{x}{\sqrt{1 + x^2}}\right) \\
& \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - \frac{x^2}{1 + x^2}}} \times \frac{1\sqrt{1 + x^2} - x \cdot \frac{1}{2}(1 + x^2)^{\frac{1}{2}} \cdot 2x}{1 + x^2} \\
& \quad \therefore \frac{dy}{dx} = \frac{\sqrt{1 + x^2}}{1} \times \frac{1 + x^2 - x^2}{(1 + x^2)^{\frac{3}{2}}} = \frac{1}{1 + x^2}.
\end{align*}

**Blunders** (-3)
- B1 Incorrect \( \sin y \)
- B2 Differentiation
- B3 Error value of \( \cos y \)
- B4 Definition of \( \sin y \) and/or \( \cos y \) (once only)
- B5 Sides of triangle (once only)
- B6 Indices

**Attempts**
- A1 Error in differentiation formula

**Worthless**
- W1 Integration
QUESTION 8

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 20 (5, 5, 5) marks Att (2, 2, 2)
Part (c) 20 (5, 5, 5) marks Att (2, 2, 2)

Part (a) 10 (5, 5) marks Att (2, 2)

8. (a)

\[ \int (2x + 3\sin 3x) \, dx \]

\[ \int 2x \, dx \quad 5 \text{ marks Att 2} \]
\[ \int \cos 3x \, dx \quad 5 \text{ marks Att 2} \]

\[ \int (2x + \cos 3x) \, dx = x^2 + \frac{1}{3} \sin 3x + \text{constant.} \]

Blunders (-3)
B1 Integration
B2 Indices
B3 No \( c \) penalise 2\textsuperscript{nd} element.

Attempts
A1 Only \( c \) correct (on 2\textsuperscript{nd} element only)

Worthless
W1 Differentiation for integration
Evaluate

(i) \[ \int_{0}^{1} 3x^2 e^{x^2} \, dx \]

(ii) \[ \int_{2}^{4} \frac{2x^3}{x^2 - 1} \, dx \]

(i) Integration 5 marks Att 2
Value 5 marks Att 2

(ii) Integration 5 marks Att 2
Value 5 marks Att 2

8 (b) (i)

\[ \int_{0}^{1} 3x^2 e^{x^2} \, dx \]

Let \( u = e^{x^2} \). \( \therefore \) \( du = 3x^2 e^{x^2} \, dx \).

\[ \therefore \int_{0}^{1} 3x^2 e^{x^2} \, dx = \left[ u \right]_{1}^{e} = e - 1. \]

OR

8 (b) (i)

\[ \int_{0}^{1} 3x^2 e^{x^2} \, dx \]

Let \( u = x^3 \)

\[ = \int e^{x^2} (3x^2 \, dx) \]

\[ = \int e^u \, du \]

\[ = e^u \]

\[ = e^{x^2} \] \( \biggr|_{0}^{1} = e^1 - e^0 = (e - 1) \)

8 (b) (ii)

\[ \int_{2}^{4} \frac{2x^3}{x^2 - 1} \, dx \]

Let \( u = x^2 - 1 \). \( \therefore \) \( du = 2x \, dx \).

\[ \int_{2}^{4} \frac{2x^3}{x^2 - 1} \, dx = \int_{3}^{15} \frac{u + 1}{u} \, du = \int_{3}^{15} \left( 1 + \frac{1}{u} \right) \, du = \left[ u + \log_e u \right]_{3}^{15} \]

\[ = 15 - 3 + \log_e 15 - \log_e 3 = 12 + \log_e 5. \]

OR
\[2 \int_{\frac{1}{2}}^{4} \frac{x^3}{x^2 - 1} \, dx = 2 \int \left[ x + \frac{x}{x^2 - 1} \right] \, dx\]

\[= 2 \left[ \int x \, dx + \int \frac{x}{x^2 - 1} \, dx \right]\]

\[= 2 \left[ \frac{x^2}{2} + \int \frac{du}{2u} \right]\]

Let \( u = x^2 - 1 \)

\[\frac{du}{dx} = 2x\]

\[\frac{du}{2} = x \, dx\]

\[= 2 \left[ \frac{x^2}{2} + \frac{1}{2} \ln u \right]\]

\[= x^2 + \ln \left( x^2 - 1 \right) \bigg|_{\frac{1}{2}}^{4}\]

\[= (16 + \ln 15) - (4 + \ln 3)\]

\[= 12 + \ln \left( \frac{15}{3} \right)\]

\[= 12 + \ln 5\]

* Incorrect substitution and unable to finish yields attempt at most.

**Blunders (-3)**

B1 Integration
B2 Indices
B3 Differentiation
B4 Limits
B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits
B8 Error logs

**Slips (-1)**

S1 Numerical
S2 Trig value
S3 \( e^0 \neq 1 \)
S4 Answer not tidied up

**Worthless**

W1 Differentiation instead of integration except where other work merits attempt.
The diagram shows the curve \( y = 4 - x^2 \) and the line \( 2x + y - 1 = 0 \).

Calculate the area of the shaded region enclosed by the curve and the line.

Points of intersection: \( 4 - x^2 = 1 - 2x \)
\( x^2 - 2x - 3 = 0 \)
\( (x - 3)(x + 1) = 0 \)
\( x = -1, \quad x = 3 \)

\[ \therefore \text{Area} = \int_{-1}^{3} [(4 - x^2) - (1 - 2x)] dx = \int_{-1}^{3} (3 + 2x - x^2) dx \]
\[ = \left[ 3x + x^2 - \frac{x^3}{3} \right]_{-1}^{3} = (9 + 9 - 9) - (-3 + 1 + \frac{1}{3}) = 10 \frac{2}{3} \]

OR
2x + y - 1 = 0 ⇒ y = 1 - 2x.
\[ y = 4 - x^2 \] ⇒ 1 - 2x = 4 - x^2
\[ \therefore x^2 - 2x - 3 = 0 \] \(\Rightarrow\) \((x - 3)(x + 1) = 0.
\[ \therefore x = 3 \text{ or } x = -1. \]
2x + y - 1 = 0 cuts x-axis at \( x = \frac{1}{2} \).
y = 4 - x^2 cuts x-axis at \( x = \pm 2 \).

Shaded region above x-axis = \( \int_{-1}^{1} \left(4 - x^2\right)dx - \int_{-1}^{1} \left(1 - 2x\right)dx \).
\[ = \left[ 4x - \frac{1}{3}x^3 \right]_{-1}^{1} - \left[ x - x^2 \right]_{-1}^{1} \]
\[ = \left( 8 - \frac{8}{3} \right) - \left( -4 + \frac{1}{3} \right) - \left[ \left( \frac{1}{2} - \frac{1}{4} \right) - (-1 - 1) \right] \]
\[ = 9 - 2 \frac{1}{4} = 6 \frac{3}{4} \]

Shaded region below x-axis = \( \int_{\frac{1}{2}}^{3} \left(1 - 2x\right)dx - \int_{2}^{3} \left(4 - x^2\right)dx \).
\[ = \left[ x - x^2 \right]_{\frac{1}{2}}^{3} - \left[ 4x - \frac{1}{3}x^3 \right]_{2}^{3} \]
\[ = \left[ 3 - 9 - \left( \frac{1}{2} - \frac{1}{4} \right) \right] - \left[ \left( 12 - 9 \right) - \left( \frac{8}{3} \right) \right] \]
\[ = - 6 \frac{1}{4} - 3 \frac{5}{1} \]
\[ = -6 \frac{1}{4} + 2 \frac{1}{3} = \frac{47}{12} \]

Total shaded region = \( \frac{27}{4} + 47 = \frac{128}{12} = \frac{32}{3} \).
Blunders (-3)
B1 Integration
B2 Indices
B3 Factors once only
B4 Calculation of point of intersection of line and curve
B5 Calculation of points where line cuts x-axis
B6 Calculation of points where curve cuts x-axis
B7 Error in area triangle
B8 Error in area formula
B9 Incorrect order in applying limits
B10 Not calculating substituted limits
B11 Error with line
B12 Error with curve
B13 Uses $\pi \int y \, dx$ for area formula

Attempts
A1 Uses volume formula
A2 Uses $y^2$ in formula

Worthless
W1 Differentiation instead of integration except where other work merits attempt
W2 Wrong area formula and no work.
MARKING SCHEME

LEAVING CERTIFICATE EXAMINATION 2008

MATHEMATICS – HIGHER LEVEL – PAPER 2

GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates’ work as follows:
   • Blunders - mathematical errors/omissions (-3)
   • Slips - numerical errors (-1)
   • Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,… etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   • any correct, relevant step in a part of a question merits at least the attempt mark for that part
   • if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   • a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,… etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The same error in the same part of a question is penalised once only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.
### QUESTION 1

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
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<tbody>
<tr>
<td>Part (b)</td>
<td>15 (5, 5, 5) marks</td>
<td>Att (2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>25 (5, 5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

#### Part (a) 10 (5, 5) marks  Att (2, 2)

1. (a) A circle with centre \((-3, 2)\) passes through the point \((1, 3)\). Find the equation of the circle.

(a) **Radius/Centre**

5 marks  Att 2

Finish 5 marks  Att 2

1. (a) Centre of circle is \(c(-3, 2)\) and \(p\) is \((1, 3)\).

\[|ap| = r = \sqrt{(-3-1)^2 + (2-3)^2} = \sqrt{16+1} = \sqrt{17}.\]

\[\therefore \text{Equation of circle: } (x+3)^2 + (y-2)^2 = 17.\]

**Blunders (-3)**
- B1 Error in distance formula
- B2 Error in circle formula

**Slips (-1)**
- S1 Uses \((-3, 2)\) as point on circle and uses \((1, 3)\) as centre

**Attempts**
- A1 Writes down correct equation of a circle and stops
Part (b) 15 (5, 5, 5) marks Att (2, 2, 2)

(b) (i) Prove that the equation of the tangent to the circle \( x^2 + y^2 = r^2 \) at the point \( (x_1, y_1) \) is \( xx_1 + yy_1 = r^2 \).

(ii) A tangent is drawn to the circle \( x^2 + y^2 = 13 \) at the point \( (2, 3) \). This tangent crosses the x-axis at \( (k, 0) \). Find the value of \( k \).

\[
\text{(b)(i) Equation } T \quad \text{5 marks} \quad \text{Finish} \quad \text{5 marks} \quad \text{Att 2}
\]

1. (b) (i)

\[
\text{Slope } op = \frac{y_1}{x_1} \Rightarrow \text{slope } T = -\frac{x_1}{y_1}.
\]

\[
\therefore \text{Equation of tangent } T: y - y_1 = -\frac{x_1}{y_1}(x - x_1).
\]

\[
\therefore yy_1 - y_1^2 = -xx_1 + x_1^2 \quad \Rightarrow \quad xx_1 + yy_1 = x_1^2 + y_1^2.
\]

But \( (x_1, y_1) \in x^2 + y^2 = r^2 \) \( \Rightarrow \quad x_1^2 + y_1^2 = r^2 \).

\[
\therefore xx_1 + yy_1 = r^2.
\]

Blunders (-3)
B1 Error in finding slope of T.
B2 Error in finding equation of tangent
B3 Error in showing \( (x_1, y_1) \in x^2 + y^2 = r^2 \) \( \Rightarrow \quad x_1^2 + y_1^2 = r^2 \).

(b) (ii) 5 marks Att 2

1. (b) (ii) Tangent at \( (2, 3) \) is \( 2x + 3y = 13 \)

\[
y = 0 \Rightarrow x = 6\frac{1}{2}. \quad \therefore k = 6\frac{1}{2}.
\]

Blunders (-3)
B1 Error in applying formula
B2 Transposition error
B3 Wrong axis

Slips (-1)
S1 Calculation errors

Attempts
A1 Correct linear formula written down with some correct substitution and stops
1. (c) A circle passes through the points \( a(8, 5) \) and \( b(9, -2) \).

The centre of the circle lies on the line \( 2x - 3y - 7 = 0 \).

(i) Find the equation of the circle.

(ii) \( p \) is a point on the major arc \( ab \) of the circle.

Show that \( |\angle apb| = 45^\circ \).

(c)(i) Two equations

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<tr>
<td>Solve</td>
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<tr>
<td>Finish</td>
<td>5 marks</td>
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</table>

(c)(i) Let circle be \( x^2 + y^2 + 2gx + 2fy + c = 0 \).

\( a(8, 5) \) \( \in \) circle \( \Rightarrow \) \( 64 + 25 + 16g + 10f + c = 0 \). \( \therefore \) \( 16g + 10f + c = -89 \).

\( b(9, -2) \) \( \in \) circle \( \Rightarrow \) \( 81 + 4 + 18g - 4f + c = 0 \). \( \therefore \) \( 18g - 4f + c = -85 \).

Centre \( (-g, -f) \) \( \in 2x - 3y - 7 = 0 \) \( \Rightarrow \) \( -2g + 3f = 7 \).

16g + 10f + c = -89

18g - 4f + c = -85

\(-2g + 14f = -4\)

But \(-2g + 3f = 7\)

\(11f = -11 \Rightarrow f = -1. \quad -2g + 3f = 7 \Rightarrow -2g = -1 \Rightarrow g = -5.\)

16g + 10f = c = -89 \( \Rightarrow \) \(-80 - 10 + c = -89 \Rightarrow c = 1.\)

\( \therefore \) Equation of circle: \( x^2 + y^2 - 10x - 2y + 1 = 0. \)

Blunders (-3)

B1 Error in finding equation.

B2 Error in formula for the equation of the circle

Slips (-1)

S1 Calculation errors

(c)(ii) Two slopes

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<td>5 marks</td>
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<td>5 marks</td>
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</table>

1. (c) (ii) Label the centre \( c \).

\( c(5, 1) \), \( a(8, 5) \), \( b(9, -2) \).

Slope \( ac = \frac{5 - 1}{8 - 5} = \frac{4}{3} \), slope \( bc = \frac{-2 - 1}{9 - 5} = \frac{-3}{4} \).

\( \frac{4}{3} \times \frac{-3}{4} = -1 \) \( \Rightarrow \) \( |\angle acb| = 90^\circ \).

But \( |\angle acb| = 2|\angle apb| \) \( \Rightarrow \) \( |\angle apb| = 45^\circ \).

Blunders (-3)

B1 Error in finding measure of angle at centre

B2 Error in finding required angle

Slips (-1)

S1 Error in calculations
### QUESTION 2

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (10, 10) marks</td>
<td>Att (3, 3)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (10, 5, 5) marks</td>
<td>Att (3, 2, 2)</td>
</tr>
</tbody>
</table>

#### 2. (a)

Given that \( 10 \vec{r} + k \vec{j} = 11 \vec{i} - 2 \vec{j} \), find the two possible values of \( k \in \mathbb{R} \).

#### 2. (a)

\[
\begin{align*}
10 \vec{r} + k \vec{j} &= 11 \vec{i} - 2 \vec{j} \\
|10 \vec{r} + k \vec{j}| &= |11 \vec{i} - 2 \vec{j}| \\
&= \sqrt{121 + 4} \\
&= \sqrt{125} \\
&= 5 \sqrt{5} \\
\therefore \sqrt{100 + k^2} &= \sqrt{125} \\
k^2 &= 25 \\
k &= \pm 5.
\end{align*}
\]

**Blunders (-3)**
- B1 Error in expression for mod of vector
- B2 Error in solving equation
- B3 One value not given.

**Slips (-1)**
- S1 Error in calculations

**Attempts**
- A1 Gives correct expression for mod of vector and stops
2. (b) \( \vec{x} = -\vec{i} + 3 \vec{j}, \quad \vec{y} = 4 \vec{i} - 2 \vec{j} \) and \( \vec{z} = \vec{x} - t \vec{y} \), where \( t \in \mathbb{R} \).

(i) Given that \( \vec{x} \perp \vec{z} \), calculate the value of \( t \).

(ii) Find the measure of \( \angle xoy \), where \( o \) is the origin.

\[
\vec{z} = \vec{x} - t \vec{y} = -\vec{i} + 3 \vec{j} - 4t \vec{i} + 2t \vec{j}
\]

\[
\therefore \vec{z} = (-1 - 4t) \vec{i} + (3 + 2t) \vec{j}.
\]

But \( \vec{x} \perp \vec{y} \Rightarrow \vec{x} \cdot \vec{y} = 0 \).

\[
\therefore \left( -\vec{i} + 3 \vec{j} \right) \left[ (-1 - 4t) \vec{i} + (3 + 2t) \vec{j} \right] = 0.
\]

\[
\therefore 1 + 4t + 9 + 6t = 0 \Rightarrow t = -1.
\]

**Blunders (-3)**

B1 Error in expressing in terms of \( \vec{i} \) and \( \vec{j} \).

B2 Error in Scalar Product property

B3 Error in solving equation

**Slips (-1)**

S1 Error in calculations

(b) (ii) 10 marks

\[
\cos \angle xoy = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| \cdot ||\vec{b}||} = \frac{\left( -\vec{i} + 3 \vec{j} \right) \left( 4 \vec{i} - 2 \vec{j} \right)}{\left( -\vec{i} + 3 \vec{j} \right) \left( 4 \vec{i} - 2 \vec{j} \right)}.
\]

\[
\cos \angle xoy = \frac{-4 - 6}{\sqrt{10} \cdot \sqrt{20}} = \frac{-10}{10 \sqrt{2}} = -\frac{1}{\sqrt{2}}.
\]

\[
\therefore \angle xoy = 135^\circ.
\]

**Blunders (-3)**

B1 Error in setting up the equation

B2 Error in solving equation

**Slips (-1)**

S1 Error in calculations
2. (c) $oabc$ is a parallelogram, where $o$ is the origin. $d$ is the midpoint of $[oa]$ and $[db]$ cuts the diagonal $[ac]$ at $p$.

(i) Given that $\vec{ap} = k \vec{ac}$, where $k \in \mathbb{R}$, express $\vec{p}$ in terms of $\vec{a}, \vec{c}$ and $k$.

(ii) Given that $\vec{bp} = l \vec{bd}$, where $l \in \mathbb{R}$, express $\vec{p}$ in terms of $\vec{a}, \vec{c}$ and $l$.

(iii) Hence find the value of $k$ and the value of $l$.

<table>
<thead>
<tr>
<th>(c) (i)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. (c) (i)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vec{ap} = k \vec{ac}$</td>
<td>$\Rightarrow$</td>
<td>$\vec{p} - \vec{a} = k \left( \vec{c} - \vec{a} \right)$</td>
</tr>
</tbody>
</table>

**Blunders (-3)**

B1 Error in simplifying $\vec{ap} = k \vec{ac}$,
B2 Error in transposing

<table>
<thead>
<tr>
<th>(c) (ii)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. (c) (ii)</td>
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</tr>
<tr>
<td>$\vec{bp} = l \vec{bd}$</td>
<td>$\Rightarrow$</td>
<td>$\vec{p} - \vec{b} = l \left( \vec{d} - \vec{b} \right)$</td>
</tr>
</tbody>
</table>

**Blunders (-3)**

B1 Error in simplifying $\vec{bp} = l \vec{bd}$,
B2 Error in finishing.
### 2. (c) (iii)

\[ \vec{p} = (1-k)\vec{a} + k\vec{c} \text{ and } \vec{p} = \left(1 - \frac{1}{2}l\right)\vec{a} + (1-l)\vec{c}. \]

\[ \therefore 1-k = 1 - \frac{1}{2}l \Rightarrow l = 2k \text{ and } k = 1-l. \]

\[ \therefore k = 1 - 2k \Rightarrow k = \frac{1}{3} \text{ and } l = \frac{2}{3}. \]

---

**Blunders (-3)**
- B1 Error in setting up equations
- B2 Error in solving equations

**Slips (-1)**
- S1 Errors in calculations
QUESTION 3

Part (a) 10 marks Att 3
Part (b) 20 (5, 5, 5, 5) marks Att (- , 2, 2, 2)
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

3. (a) The parametric equations \( x = 7t - 4 \) and \( y = 3 - 3t \) represent a line, where \( t \in \mathbb{R} \).

Find the Cartesian equation of the line.

\[
3. (a) \quad \begin{align*}
x &= 7t - 4 \quad \Rightarrow \quad 3x &= 21t - 12 \\
y &= 3 - 3t \quad \Rightarrow \quad 7y &= 21 - 21t \\
\therefore \quad 3x + 7y &= 9.
\end{align*}
\]

Blunders (-3)
B1 Error in setting up equations
B2 Error in solving the equations

Slips (-1)
S1 Errors in calculations

Part (b) 20 (5, 5, 5, 5) marks Att (- , 2, 2, 2)

3. (b) \( a (2, 1), \ b (10, 7), \ c (14, 10) \) and \( d (7, 1) \) are four points.

(i) Plot \( a, b, c \) and \( d \) on the co-ordinate plane.

(ii) Verify that \( |ab| = 2|bc| \) and \( |ab| = 2|ad| \).

(iii) Find \( a', b', c' \) and \( d' \), the respective images of \( a, b, c \) and \( d \) under the transformation \( f : (x, y) \rightarrow (x', y') \), where \( x' = x + y \) and \( y' = x - 2y \).

(iv) Verify that \( |a'b'| = 2|b'c'| \) but that \( |a'b'| \neq 2|a'd'| \).

(b) (i) 5 marks Hit / Miss

3. (b) (i) \( y \)-axis \( \uparrow \)

\( c \) (14, 10)

\( b \) (10, 7)

\( a \) (2, 1)

\( d \) (7, 1)

\( x \)-axis

* All four points correct: 5 marks. Otherwise, 0 marks.
(b) (ii) 5 marks

\[ a(2,1), b(10,7), c(14,10), \ d(7,1). \]

\[ \left| ab \right| = \sqrt{(10 - 2)^2 + (7 - 1)^2} = \sqrt{64 + 36} = 10. \]

\[ \left| bc \right| = \sqrt{(10 - 14)^2 + (7 - 10)^2} = \sqrt{16 + 9} = 5. \]

\[ \therefore \left| ab \right| = 2\left| bc \right|. \]

\[ \left| ad \right| = \sqrt{(2 - 7)^2 + (1 - 1)^2} = \sqrt{25 + 0} = 5. \]

\[ \therefore \left| ab \right| = 2\left| ad \right|. \]

**Blunders (-3)**

B1  Error in distance formula

B2  Incorrect squaring

**Slips (-1)**

S1  Error in calculations

---

3. (b) (iii) 5 marks

\[ a(2,1), b(10,7), c(14,10), \ d(7,1). \]

\[ a' = f(2,1) \Rightarrow a' = (3,0). \]

\[ b' = f(10,7) \Rightarrow b' = (17, -4). \]

\[ c' = f(14,10) \Rightarrow c' = (24, -6). \]

\[ d' = f(7,1) \Rightarrow d' = (8, 5). \]

**Blunders (-3)**

B1  Any error in finding images

---

3. (b) (iv) 5 marks

\[ a'(2,1), b'(10,7), c'(14,10), \ d'(7,1). \]

\[ a' = f(2,1) \Rightarrow a' = (3,0). \]

\[ b' = f(10,7) \Rightarrow b' = (17, -4). \]

\[ c' = f(14,10) \Rightarrow c' = (24, -6). \]

\[ d' = f(7,1) \Rightarrow d' = (8, 5). \]

\[ \left| a'b' \right| = \sqrt{(3 - 17)^2 + (0 + 4)^2} = \sqrt{196 + 16} = \sqrt{212} = 2\sqrt{53}. \]

\[ \left| b'c' \right| = \sqrt{(17 - 24)^2 + (-4 + 6)^2} = \sqrt{49 + 4} = \sqrt{53}. \]

\[ \left| a'd' \right| = \sqrt{(3 - 8)^2 + (0 - 5)^2} = \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}. \]

\[ \therefore \left| a'b' \right| = 2\left| b'c' \right| \text{ but } \left| a'b' \right| \neq 2\left| a'd' \right|. \]

**Blunders (-3)**

B1  Error in distance formula

B2  Incorrect squaring

**Slips (-1)**

S1  Error in calculations
3.(c) Prove that the perpendicular distance from the point \((x_1, y_1)\) to the line \(ax + by + c = 0\) is \(\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}\).

(c) Diagram
Area of \(pqr\)  5 marks  Att 2
Area of image  5 marks  Att 2
Solves equation  5 marks  Att 2

\[
\text{Area triangle } pqr = \frac{1}{2} |qr||ps| = \frac{1}{2} \left| \frac{c^2}{a^2} + \frac{c^2}{b^2} \right| |ps| = \frac{1}{2} \left| \frac{c}{ab} \right| \sqrt{a^2 + b^2} \cdot |ps|.
\]

Translating \(q\left(0, -\frac{c}{b}\right)\) to \((0, 0)\) \(\Rightarrow\) \(p(x_1, y_1) \rightarrow \left(x_1, y_1 + \frac{c}{b}\right)\) and \(r\left(-\frac{c}{a}, 0\right) \rightarrow \left(-\frac{c}{b}, 0\right)\).

\[
\therefore \text{Area triangle } pqr = \frac{1}{2} \left| x_1 \left(\frac{c}{b}\right) - \left(-\frac{c}{a}\right) \left(y_1 + \frac{c}{b}\right) \right| = \frac{1}{2} \left| \frac{c}{b} x_1 + \frac{c}{a} y_1 + \frac{c^2}{ab} \right| = \frac{1}{2} \left| \frac{c}{ab} \right| \left| ax_1 + by_1 + c \right|.
\]

\[
\therefore \sqrt{a^2 + b^2} \cdot |ps| = |ax_1 + by_1 + c| \Rightarrow |ps| = \text{distance} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.
\]

Blunders (-3)
B1 Error in diagram
B2 Error in area each time
B3 Error in setting up or solving equation

Slips (-1)
S1 Error in calculations
QUESTION 4

Part (a) 10 marks

4. (a) $A$ and $B$ are acute angles such that $\tan A = \frac{5}{12}$ and $\tan B = \frac{3}{4}$.

Find $\cos(A - B)$, as a fraction.

\[ \tan A = \frac{5}{12} \quad 5 \quad 13 \quad A \quad 90^0 \quad 12 \]

\[ \tan B = \frac{3}{4} \quad 3 \quad 5 \quad B \quad 90^0 \quad 4 \]

\[ \therefore \sin A = \frac{5}{13}, \quad \cos A = \frac{12}{13}. \]

\[ \therefore \sin B = \frac{3}{5}, \quad \cos B = \frac{4}{5}. \]

\[ \therefore \cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{12}{13} \cdot \frac{4}{5} + \frac{5}{13} \cdot \frac{3}{5} = \frac{63}{65}. \]

Blunders (-3)
B1 Error in finding $\sin A$ or $\cos A$ or $\sin B$ or $\cos B$ each time
B2 Sign error in $\cos(A - B)$

Slips (-1)
S1 Error in calculations

Attempts
A1 Draws a right angled triangle with length of one side indicated
A2 Evaluates $A$ and $B$ and subtracts
4. (b) (i) Show that \[
\frac{\sin 2A}{1 + \cos 2A} = \tan A.
\]
(ii) Hence, or otherwise, prove that \(\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1\).

(b) (i) 10 marks

\[
\frac{\sin 2A}{1 + \cos 2A} = \frac{2 \sin A \cos A}{2 \cos^2 A} = \frac{\sin A}{\cos A} = \tan A.
\]

**Blunders (-3)**
B1 Error in simplifying \(\sin 2A\) or \(1 + \cos 2A\)
B2 Error in finding \(\tan A\).

(b) (ii) 10 marks

\[
\tan 22\frac{1}{2}^\circ = \frac{\sin 45^\circ}{1 + \cos 45^\circ} = \frac{1}{1 + \frac{\sqrt{2}}{2}} = \frac{1}{\frac{\sqrt{2} + 2}{2}}
\]
\[
= \frac{1}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \frac{\sqrt{2} - 1}{2 - 1} = \sqrt{2} - 1.
\]

**Blunders (-3)**
B1 Fails to link double and half angle correctly
B2 Error in evaluation
B3 Answer not in required form
Part (c) 20 (5, 5, 5, 5) marks Att (2, - , - , - )

(c) In the triangle $pqr$, $\angle rsq = \theta^\circ$, $\angle prs = \alpha^\circ$, $|rq| = 1$, $|ps| = 1$ and $|sq| = 1$.

(i) Find $|sr|$ in terms of $\theta$.

(ii) Hence, or otherwise, show that $\tan \theta = 3 \tan \alpha$.

---

(c) (i) 5 marks Att 2

4 (c) (i)

In triangle $qrs$, $\angle srq = \theta$ as $|sq| = |qr| \therefore \angle sqr = 180^\circ - 2\theta$.

\[
\frac{|sr|}{\sin(180^\circ - 2\theta)} = \frac{1}{\sin \theta}
\]

\[
\Rightarrow |sr| = \frac{\sin 2\theta}{\sin \theta} = \frac{2\sin \theta \cos \theta}{\sin \theta} = 2 \cos \theta.
\]

Blunders (-3)
B1 Error in applying sine rule/cosine rule each time.
B2 Error in simplifying $\sin(180^\circ - 2\theta)$ or $\cos(180^\circ - 2\theta)$
B3 Error in solving equation each time

Slips (-1)
S1 Error in calculations

(c)(ii) Set up Sine Rule 5 marks Hit/Miss
Expand $\sin(\theta - \alpha)$ 5 marks Hit/Miss
Finish 5 marks Hit/Miss

4. (c) (ii)

In the triangle $psr$, $\angle rps = \theta - \alpha$.

\[
\therefore \frac{\sin(\theta - \alpha)}{2\cos \theta} = \frac{\sin \alpha}{1} \Rightarrow \sin(\theta - \alpha) = 2 \cos \theta \sin \alpha.
\]

\[
\therefore \sin \theta \cos \alpha - \cos \theta \sin \alpha = 2 \cos \theta \sin \alpha \Rightarrow \sin \theta \cos \alpha = 3 \cos \theta \sin \alpha
\]

Dividing across by $\cos \theta \cos \alpha$ results in

\[
\frac{\sin \theta}{\cos \theta} = \frac{3 \sin \alpha}{\cos \alpha},
\]

\[
\therefore \tan \theta = 3 \tan \alpha.
\]

* Second 5 marks only available if first 5 has been awarded.
Third 5 marks only available if second 5 has been awarded.
QUESTION 5

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 20 (5, 15) marks Att (-, 5)
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

Part (a) 10 (5, 5) marks Att (2, 2)

5. (a) In the shaded sector in the diagram, the arc is 6 cm long, and the angle of the sector 0.75 radians. Find the area of the sector.

(a) Radius 5 marks Att 2
Area 5 marks Att 2

5. (a) Length of arc = rθ ⇒ r(0.75) = 6 ⇒ r = 8 cm.

Area of sector = \( \frac{1}{2} r^2 \theta = \frac{1}{2} \times 64 \times (0.75) = 24 \text{ cm}^2 \).

Blunders (-3)
B1 Error in calculating radius
B2 Error in calculating area

Slips (-1)
S1 No units

Attempts
A1 Correct formula and some correct substitution and stops

Part (b) 20 (5, 15) marks Att (-, 5)

(b) (i) Express \( \sin 4x - \sin 2x \) as a product.
(ii) Find all the solutions of the equation \( \sin 4x - \sin 2x = 0 \) in the domain \( 0^\circ \leq x \leq 180^\circ \).

(b) (i) 5 marks Hit / Miss

5. (b) (i) \( \sin 4x - \sin 2x = 2\cos 3x \sin x \).

(b) (ii) 15 marks Att 5

5. (b) (ii) \[
\sin 4x - \sin 2x = 0 \Rightarrow 2\cos 3x \sin x = 0.
\]
\[
\therefore \cos 3x = 0 \quad \text{or} \quad \sin x = 0.
\]
\[
\therefore 3x = 90^\circ, 270^\circ, 450^\circ \quad \text{or} \quad x = 0^\circ, 180^\circ.
\]
\[
\therefore x = 30^\circ, 90^\circ, 150^\circ \quad \text{or} \quad x = 0^\circ, 180^\circ.
\]
Solution is \{0\degree, 30\degree, 90\degree, 150\degree, 180\degree\}.

Blunders (-3)
B1 Error in solving equation
B2 Solutions omitted
(c) A triangle has sides of lengths $a$, $b$ and $c$. The angle opposite the side of length $a$ is $A$.

(i) Prove that $a^2 = b^2 + c^2 - 2bc \cos A$.

(ii) If $a$, $b$ and $c$ are consecutive whole numbers, show that

$$\cos A = \frac{a + 5}{2a + 4}.$$

(c)(i) Diagrams 5 marks Att 2
Value for $l$ 5 marks Att 2
Finish 5 marks Att 2

5 (c) (i) case 1: A obtuse

\[ k = b \cos(180° - A) = -b \cos A \]
\[ \therefore l = c + k = c - b \cos A \]
\[ h = b \sin(180° - A) = b \sin A \]

Both cases continue:

by Pythagoras’ theorem: $a^2 = h^2 + l^2$

\[ a^2 = (b \sin A)^2 + (c - b \cos A)^2 \]
\[ a^2 = b^2 \sin^2 A + c^2 + b^2 \cos^2 A - 2bc \cos A. \]
\[ a^2 = b^2 (\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A \]
\[ a^2 = b^2 + c^2 - 2bc \cos A. \]

* Correct acute case but omits or mishandles obtuse case, or vice versa: one blunder.

Blunders (-3)
B1 Error in diagram(s)
B2 Error in finding $l$
B3 Error in finishing

Attempts
A1 Draws diagram with correct labelling.

Or
(c)(i) Diagram  5 marks  Att 2
Substitutes in formula  5 marks  Att 2
Finish  5 marks  Att 2

(covering all cases)

\[
\begin{align*}
\text{Distance formula } & \Rightarrow a = \sqrt{(c - b \cos A)^2 + (0 - b \sin A)^2} \\
& = \sqrt{c^2 - 2bc \cos A + b^2 \cos^2 A + b^2 \sin^2 A} \\
& = \sqrt{c^2 - 2bc \cos A + b^2 (\cos^2 A + \sin^2 A)} \\
& = b^2 + c^2 - 2bc \cos A.
\end{align*}
\]

Blunders (-3)
B1 Error in diagram
B2 Error in use of distance formula
B3 Error in finishing

Attempts
A1 Draws diagram with correct labelling.

(c) (ii)  5 marks  Att 2

5. (c) (ii) \(a, b\) and \(c\) are consecutive whole numbers,
\(\therefore b = a + 1\) and \(c = a + 2\).

\[
\begin{align*}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \Rightarrow \cos A = \frac{(a+1)^2 + (a+2)^2 - a^2}{2(a+1)(a+2)}.
\end{align*}
\]

\[
\begin{align*}
\cos A &= \frac{a^2 + 2a + 1 + a^2 - a^2 + 4a + 4 - a^2}{2(a+1)(a+2)} = \frac{a^2 + 6a + 5}{2(a+1)(a+2)} \\
&= \frac{(a+1)(a+5)}{2(a+1)(a+2)} = \frac{a+5}{2a+4}.
\end{align*}
\]

Blunders (-3)
B1 Numbers not consecutive
B2 Error in substitution
B3 Error in simplification

Slips (-1)
S1 Error in calculations
QUESTION 6

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20(10, 5, 5) marks</td>
<td>Att (3, 2, 2)</td>
</tr>
</tbody>
</table>

6. (a) In a certain subject, the examination consists of a project, a practical test, and a written paper. The overall mark is the weighted mean of the percentages achieved in these three components, using the weights 2, 3 and 5, respectively.

Michael scores 65% in the project and 80% in the practical.

What percentage mark must he get in the written paper in order to get an overall result of 70%?

(a) 10 marks Att 3

6. (a) Let Michael require \(x\)% on written paper.

\[
\text{Weighed mean} = \frac{2(65) + 3(80) + 5(x)}{2 + 3 + 5} = \frac{370 + 5x}{10} = 70.
\]

\[
370 + 5x = 700 \Rightarrow 5x = 330 \Rightarrow x = 66.
\]

\(66\%\) required on written paper.

**Blunders (-3)**

B1 Error in using weights
B2 Error in setting up the equation
B3 Error in solving equation

**Slips (-1)**

S1 Error in calculations

**Attempts**

A1 Tries to find the mean
### 6 (b) Solve the difference equation \( u_{n+2} - 4u_{n+1} + u_n = 0 \), where \( n \geq 0 \), given that \( u_0 = 1 \) and \( u_1 = 2 \).

**Form quadratic**

**Solve quadratic**

**General Term**

**Finish**

\[
\begin{align*}
6 \ (b) \quad u_{n+2} - 4u_{n+1} + u_n &= 0. \\
\therefore \ x^2 - 4x + 1 &= 0 \quad \Rightarrow \quad x = \frac{4 \pm \sqrt{16 - 4}}{2} \quad \Rightarrow \quad x = \frac{4 \pm 2\sqrt{3}}{2} \quad \Rightarrow \quad x = 2 \pm \sqrt{3}.
\end{align*}
\]

\[
\begin{align*}
u_n &= l(\alpha)^n + m(\beta)^n = \frac{l(2 + \sqrt{3})^n}{2} + \frac{m(2 - \sqrt{3})^n}{2}.
\end{align*}
\]

\[
\begin{align*}
u_0 &= 1 \quad \Rightarrow \quad l + m = 1 \quad \text{and} \quad u_1 = 2 \quad \Rightarrow \quad l(2 + \sqrt{3}) + m(2 - \sqrt{3}) = 2.
\end{align*}
\]

\[
\begin{align*}
\therefore \quad 2(l + m) + \sqrt{3}(l - m) &= 2 \quad \Rightarrow \quad 2 + \sqrt{3}(1 - m - m) = 2 \\
\Rightarrow \quad \sqrt{3}(1 - 2m) &= 0 \quad \Rightarrow \quad m = \frac{1}{2} \quad \text{and} \quad l = \frac{1}{2}.
\end{align*}
\]

\[
\begin{align*}
\therefore \quad u_n &= \frac{1}{2} \left[ \left(2 + \sqrt{3}\right)^n + \left(2 - \sqrt{3}\right)^n \right].
\end{align*}
\]

**Blunders (−3)**

B1 Error in setting up quadratic

B2 Error in solving quadratic

B3 Error in finding General Term

B4 Error in finding \( l \) and \( m \)

**Slips (−1)**

S1 Error in calculations
A bag contains discs of three different colours. There are 5 red discs, 1 white disc and $x$ black discs. Three discs are picked together at random.

(i) Write down an expression in $x$ for the probability that the three discs are all different in colour.

(ii) If the probability that the three discs are all different in colour is equal to the probability that they are all black, find $x$.

**Solution:**

(i) 
\[
P(\text{three discs different in colour}) = \frac{5 \times 1 \times x}{(6+x)^3} = \frac{5x}{(6+x)(5+x)(4+x)}\]

(ii) 
\[
P(\text{three black discs}) = \frac{x(x-1)(x-2)}{(6+x)^3} = \frac{x(x-1)(x-2)}{(6+x)(5+x)(4+x)}\]

\[
\therefore \frac{x(x-1)(x-2)}{(6+x)(5+x)(4+x)} = \frac{30x}{(6+x)(5+x)(4+x)}\]

\[
\therefore (x-7)(x+4) = 0 \Rightarrow x = 7 \text{ as } x \neq -4.\]

7 black discs.

**Blunders (-3)**
B1 Error in numerator
B2 Error in denominator

**Slips (-1)**
S1 Error in calculations
QUESTION 7

Part (a)  10 (5, 5) marks  Att ( - , 2)
Part (b)  20 (10, 5, 5) marks  Att (3, 2, 2)
Part (c)  20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

7. (a) Katie must choose five subjects from nine available subjects.
    The nine subjects include French and German.
    (i) How many different combinations of five subjects are possible?
    (ii) How many different combinations are possible if Katie wishes to
         study German but not French?

(a) (i)  5 marks  Hit / Miss

7. (a) (i) Number of combinations = \(^9C_5 = 126.\)

(a) (ii)  5 marks  Att 2

7. (a) (ii) Number of combinations = \(^7C_4 = 35\)

Blunders (-3)
B1 Error in \(n\) or \(r\). (i.e., \(n \neq 7\) or \(r \neq 4\)).
B2 Error in evaluation of \(^7C_4\)

Slips (-1)
S1 Error in calculations

Part (b)  20 (10, 5, 5) marks  Att (3, 2, 2)

7. (b) (b) Four cards are drawn together from a pack of 52 playing cards.
    Find the probability that
    (i) the four cards drawn are the four aces
    (ii) two of the cards are clubs and the other two are diamonds
    (iii) there are three clubs and two aces among the four cards.

(b)(i)  10 marks  Att 3

7. (b) (i)
Probability (four aces) = \(\frac{^4C_4}{^52C_4} = \frac{1}{270725}\).

Blunders (-3)
B1 Incorrect total possible
B2 Incorrect total favourable

Slips (-1)
S1 Errors in calculations
7. (b) (ii) Probability (2 clubs and 2 diamonds) = \( \frac{\binom{13}{2} \times \binom{13}{2}}{\binom{52}{4}} = \frac{78 \times 78}{270725} = \frac{6084}{270725} \).

Blunders (-3)
B1 Incorrect total possible
B2 Incorrect total favourable

Slips (-1)
S1 Errors in calculations

7. (b) (iii) Probability = \( \frac{1 \times \binom{12}{2} \times \binom{3}{1}}{\binom{52}{4}} = \frac{198}{270725} \).

Blunders (-3)
B1 Incorrect total possible
B2 Incorrect total favourable

Slips (-1)
S1 Errors in calculations
7. (c) (i) The arithmetic mean of the three numbers \( x_1, x_2, x_3 \) is \( \bar{x} \).

Let \( d_1 = x_1 - \bar{x}, \) \( d_2 = x_2 - \bar{x} \) and \( d_3 = x_3 - \bar{x} \).

Show that \( \sum_{r=1}^{3} d_r = 0 \).

(ii) The standard deviation of the three numbers \( x_1, x_2, x_3 \) is \( \sigma \).

Given any real number \( b \), let \( k^2 = \sum_{r=1}^{3} \frac{(d_r - b)^2}{3} \).

Show that \( \sigma^2 = k^2 - b^2 \).

(i) Mean

Show \( \sum d = 0 \)

7. (c) (i)

\[
\bar{x} = \frac{x_1 + x_2 + x_3}{3} \Rightarrow x_1 + x_2 + x_3 = 3\bar{x}.
\]

\[
\sum_{r=1}^{3} d_r = d_1 + d_2 + d_3 = x_1 - \bar{x} + x_2 - \bar{x} + x_3 - \bar{x} = x_1 + x_2 + x_3 - 3\bar{x} = 0.
\]

(iii) Expression for \( k^2 \)

Finish \( k^2 \)

7. (c) (ii)

\[
k^2 = \frac{3}{3} \sum_{r=1}^{3} \frac{(d_r - b)^2}{3} = \frac{(d_1 - b)^2 + (d_2 - b)^2 + (d_3 - b)^2}{3}
\]

\[
= \frac{(x_1 - \bar{x} - b)^2 + (x_2 - \bar{x} - b)^2 + (x_3 - \bar{x} - b)^2}{3}
\]

\[
= \frac{(x_1 - \bar{x})^2 - 2b(x_1 - \bar{x}) + b^2 + (x_2 - \bar{x})^2 - 2b(x_2 - \bar{x}) + b^2 + (x_3 - \bar{x})^2 - 2b(x_3 - \bar{x}) + b^2}{3}
\]

\[
= \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}{3} \frac{2b[(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x})]}{3} + b^2.
\]

\[
= \sigma^2 - 0 + b^2.
\]

\[
\therefore k^2 = \sigma^2 + b^2 \Rightarrow \sigma^2 = k^2 - b^2.
\]
8. **(a)** Use the ratio test to show that \( \sum_{n=1}^{\infty} \frac{2^{3n+1}}{n!} \) is convergent.

\[
\frac{u_{n+1}}{u_n} = \frac{2^{3n+4}}{(n+1)!} \times \frac{n!}{2^{3n+1}}
\]

\[
\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \frac{2^{3n+4}}{(n+1)!} \times \frac{n!}{2^{3n+1}} = \lim_{n \to \infty} \frac{2^3}{n+1} = 0 < 1. \quad \therefore \text{Convergent.}
\]

**Blunders (-3)**

B1 Error in expressing \( u_{n+1} = \frac{2^{3n+4}}{(n+1)!} \).

B2 Error in stating Ratio Test

B3 Error in evaluating limit
### Part (b)

**20 (10, 10) marks**

**Att (3, 3)**

**8 (b) pqr is an equilateral triangle of side 6 cm.**

\( \triangle abc \) is a rectangle inscribed in the triangle as shown.

\[ ab = x \text{ cm and } bc = y \text{ cm.} \]

(i)  Express \( y \) in terms of \( x \).

(ii) Find the maximum possible area of \( abcd \).

---

**8 (b) (i) 10 marks**

\[ pqr = 6 \text{ and } ab = x \Rightarrow bq = 3 - \frac{1}{2}x. \]

\[ \angle cqb = 60^\circ \text{. tan} \angle cqb = \frac{bc}{bq} \Rightarrow \frac{y}{3 - \frac{1}{2}x} = \tan 60^\circ. \]

\[ \therefore y = \left(3 - \frac{1}{2}x\right)\sqrt{3} \text{ cm.} \]

---

**Blunders (-3)**

B1 Fails to express \( bq \) in terms of \( x \).

B2 Error in trig ratio or in use of similar triangles

B3 Error in setting up the equation

---

**8 (b) (ii) 10 marks**

Area \( abcd = A = xy = x\sqrt{3}\left(3 - \frac{1}{2}x\right) \]

\[ A = 3\sqrt{3}x - \frac{1}{2}\sqrt{3}x^2. \]

\[ \therefore \frac{dy}{dx} = 3\sqrt{3} - \sqrt{3}x = 0 \text{ for maximum area. } \therefore x = 3. \]

For \( x = 3 \), \[ \frac{d^2y}{dx^2} = -\sqrt{3} < 0 \Rightarrow \text{ maximum.} \]

\[ A = 3\sqrt{3}\left(3 - \frac{3}{2}\right) = \frac{9\sqrt{3}}{2} \text{ cm}^2. \]

* Incorrect \( y \) from part (i) ⇒ attempt at most for part (ii)

---

**Blunders (-3)**

B1 Error in expression for area

B2 Error in differentiation

B3 Error in solving equation

B4 Does not find area

---

**Slips (-1)**

S1 Errors in calculations

---

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Part (c) 20 (10, 5, 5) marks Att (3, 2, 2)

8 (c) (i) Derive the Maclaurin series for \( f(x) = \cos x \), up to and including the term containing \( x^4 \).

(ii) Hence, or otherwise, show that the first three non-zero terms of the Maclaurin series for \( f(x) = \cos^2 x \) are \( 1 - x^2 + \frac{x^4}{3} \).

(iii) Use these to find an approximation for \( \cos^2(0.2) \), giving your answer correct to four decimal places.

---

(c) (i) 10 marks Att 3

\[
\begin{align*}
\text{8 (c) (i)} & \\
& f(x) = \cos x = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \ldots \\
& f(0) = \cos 0 = 1. \\
& f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0. \\
& f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1. \\
& f'''(x) = \sin x \Rightarrow f'''(0) = \sin 0 = 0. \\
& f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(0) = \cos 0 = 1. \\
\therefore f(x) = \cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots
\end{align*}
\]

Blunders (-3)
B1 Incorrect differentiation
B2 Incorrect evaluation of \( f^{(4)}(0) \)
B3 Each term not derived
B4 Error in Maclaurin Series

Slips (-1)
S1 Error in calculations
### (c) (ii)

<table>
<thead>
<tr>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>8 (c) (ii)</strong></td>
<td></td>
</tr>
<tr>
<td>[ \cos^2 x = \frac{1}{2} (1 + \cos 2x) = \frac{1}{2} \left( 1 + 1 - \frac{4x^2}{2} + \frac{16x^4}{24} \right) = \frac{1}{2} \left( 2 - 2x^2 + \frac{2x^4}{3} \right), ]</td>
<td></td>
</tr>
<tr>
<td>∴ [ \cos^2 x = 1 - x^2 + \frac{x^4}{3}. ]</td>
<td></td>
</tr>
</tbody>
</table>

**Blunders (-3)**
- B1 Error in trig or multiplication

**Slips (-1)**
- S1 Errors in calculations

### (c) (iii)

<table>
<thead>
<tr>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>8 (c) (iii)</strong></td>
<td></td>
</tr>
<tr>
<td>[ \cos^2 x = 1 - x^2 + \frac{x^4}{3}. ]</td>
<td></td>
</tr>
<tr>
<td>[ \Rightarrow \cos^2 (0.2) = 1 - 0.04 + 0.00053 = 0.96053 = 0.9605. ]</td>
<td></td>
</tr>
</tbody>
</table>

**Blunders (-3)**
- B1 Error in terms

**Slips (-1)**
- S1 Error in calculations
QUESTION 9

Part (a) 10 marks Att 3

Part (b) 20 (10, 5, 5) marks Att (3, 2, 2)

Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

---

**Part (a) 10 marks Att 3**

**9(a)**

20% of the items produced by a machine are defective. Four items are chosen at random. Find the probability that none of the chosen items is defective.

(a) 10 marks Att 3

9 (a) Probability (one defective) = \( p = \frac{1}{5} \); probability (one not defective) = \( 1 - p = q = \frac{4}{5} \).

Probability (four not defective) = \( ^4C_4 \left( \frac{1}{5} \right)^0 \left( \frac{4}{5} \right)^4 = \frac{256}{625} \).

Blunders (-3)
B1 Incorrect \( p \) or \( q \).
B2 Error in Binomial

---

**Part (b) 20 (10, 5, 5) marks Att (3, 2, 2)**

9 (b) Anne and Brendan play a game in which they take turns throwing a die. The first person to throw a six wins. Anne has the first throw.

(i) Find the probability that Anne wins on her second throw.

(ii) Find the probability that Anne wins on her first, second or third throw.

(iii) By finding the sum to infinity of a geometric series, or otherwise, find the probability that Anne wins the game.

(b) (i) 10 marks Att 3

9 (b) (i)

Probability (Anne wins on second throw) = \( P \) (Anne loses on 1\textsuperscript{st} throw) \cdot P \) (Brendan loses on 1\textsuperscript{st} throw) \cdot P \) (Anne wins on 2\textsuperscript{nd} throw) = \( \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{25}{216} \).

Blunders (-3)
B1 Any extra throw included or each incorrect prob
(b) (ii) 5 marks

9 (b) (ii)

| Probability (Anne wins on 3rd throw) | = | \(\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{625}{7776}\). |
| Probability (Anne wins on 1st, 2nd or 3rd throw) | = | \(\frac{1}{6} + \frac{25}{216} + \frac{625}{7776} = \frac{2821}{7776}\). |

Blunders (-3)
B1 Each probability omitted

Slips (-1)
S1 Errors in calculations

(b) (iii) 5 marks

9 (b) (iii)

\[ p = \text{probability(Anne wins game)} = \frac{1}{6} + \frac{25}{216} + \frac{625}{7776} + \ldots \ldots \ldots \]

i.e. the sum to infinite of a geometric series where \(a = \frac{1}{6}\) and \(r = \frac{25}{36}\).

\[ p = \frac{a}{1 - r} = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{\frac{1}{6}}{\frac{11}{36}} = \frac{6}{11}. \]

Or

\[ p = P(\text{Anna eventually wins}) \]
\[ = P(\text{person whose turn is next eventually wins}) \]
\[ P(\text{Ann wins}) + P(\text{Brendan wins}) = 1 \]
\[ p + \frac{5}{6}p = 1 \]
\[ \frac{11}{6}p = 1 \]
\[ p = \frac{6}{11} \]

Blunders (-3)
B1 Error in \(a\) or \(r\)
B2 Error in sum to infinity

Slips (-1)
S1 Errors in calculations
In order to test the hypothesis that a particular coin is unbiased, the coin is tossed 400 times. The number of heads observed is $x$. Between what limits should $x$ lie in order that the hypothesis not be rejected at the 5% significance level?

(c) Find $\mu$

Find $\sigma$

Standard units

Conclusion

\[
n = 400, \quad p = \frac{1}{2}, \quad q = \frac{1}{2}.
\]
\[
\mu = np = 200 \quad \text{and} \quad \sigma = \sqrt{npq} = 10.
\]
\[
-1.96 \leq z \leq 1.96 \quad \Rightarrow \quad -1.96 \leq \frac{x - 200}{\frac{10}{10}} \leq 1.96.
\]
\[
\therefore -19.6 \leq x - 200 \leq 19.6 \quad \Rightarrow \quad 180.4 \leq x \leq 219.6.
\]
\[
\therefore 181 \leq x \leq 219.
\]

Blunders (-3)

B1 Error in finding mean
B2 Error in finding standard deviation
B3 Error in units
B4 Error in conclusion

Slips (-1)

S1 Errors in calculations
QUESTION 10

Part (a) 20 (5, 5, 10) marks Att (-, - , 3)
Part (b) 30 (10, 10, 5, 5) marks Att (3, 3, 2, 2)

10 (a) Let $x \oplus y = x + y - 4$, where $x, y \in \mathbb{Z}$.

(i) Find the identity element.

(ii) Find the inverse of $x$.

(iii) Determine whether $\oplus$ is associative on $\mathbb{Z}$.

(a) (i) 5 marks Hit / miss

10 (a) (i)

\[
x \oplus e = x + e - 4 = x.
\]

\[
\therefore e = 4.
\]

(a) (ii) 5 marks Hit / miss

10 (a) (ii)

\[
x \oplus x^{-1} = e = 4.
\]

\[
\therefore x + x^{-1} - 4 = 4 \Rightarrow x^{-1} = 8 - x.
\]

(a) (iii) 10 marks Att 3

10 (a) (iii)

If associative then:

\[
(x \oplus y) \oplus z = x \oplus (y \oplus z),
\]

\[
(x + y - 4) \oplus z = x \oplus (y + z - 4)
\]

\[
(x + y - 4) + z - 4 = x + (y + z - 4) - 4
\]

\[
x + y + z - 8 = x + y + z - 8, \text{ as } + \text{ is associative on } \mathbb{Z}.
\]

\[
\therefore \text{Operation } \oplus \text{ is associative.}
\]

Blunders (-3)

B1 Error in defining associativity.

B2 Error in applying rule

B3 No conclusion

Slides (-1)

S1 Error in calculations
Part (b)  30 (10, 10, 5, 5) marks  Att (3, 3, 2, 2)

10 (b)  

\((A, \circ)\) and \((B, \ast)\) are two groups. \(A = \{k, l, m, n\}\) and \(B = \{p, q, r, s\}\), and the Cayley tables for \((A, \circ)\) and \((B, \ast)\) are shown.

<table>
<thead>
<tr>
<th></th>
<th>(k)</th>
<th>(l)</th>
<th>(m)</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\circ)</td>
<td>(k)</td>
<td>(l)</td>
<td>(m)</td>
<td>(n)</td>
</tr>
</tbody>
</table>

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<tr>
<th></th>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
<th>(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\ast)</td>
<td>(p)</td>
<td>(q)</td>
<td>(r)</td>
<td>(s)</td>
</tr>
</tbody>
</table>

(i) Write down the identity element of \((A, \circ)\) and hence find a generator of \((A, \circ)\).

(ii) Find the order of each element in \((B, \ast)\).

(iii) Give an isomorphism \(\phi\) from \((A, \circ)\) to \((B, \ast)\), justifying fully that it is an isomorphism.

(b) (i)  10 marks  Att 3

10 (b) (i)

\(l\) is the identity of \((A, \circ)\).

\(m^1 = m, m^2 = k, m^3 = n, m^4 = l. \therefore m\) is a generator. \(n\) is also a generator.

Blunders (-3)
B1 Error in selecting Identity
B2 Error in verifying generator

(b) (ii)  10 marks  Att 3

10 (b) (ii)

\(r\) is the identity of \((B, \ast)\). \(r\) is of order 1.

\(p^2 = r \implies p\) is of order 2.

\(q^4 = r \implies q\) is of order 4.

\(s^4 = r \implies s\) is of order 4.

Blunders (-3)
B1 Error in order each time
(iii) Isomorphism 5 marks
Justify 5 marks

10 (b) (iii)

\[(A, \circ) \to (B, \ast)\]

Justification:

\[l\) is the identity of \((A, \circ)\) and \(r\) is the identity of \((B, \ast)\).

Products involving the identity will clearly carry across. Others are:

\[\phi(k \circ k) = \phi(l) = r\) and \(\phi(k) \ast \phi(k) = p \ast p = r.\]
\[\phi(k \circ m) = \phi(n) = s\) and \(\phi(k) \ast \phi(m) = p \ast q = s.\]
\[\phi(k \circ n) = \phi(m) = q\) and \(\phi(k) \ast \phi(n) = p \ast s = q.\]
\[\phi(m \circ m) = \phi(k) = p\) and \(\phi(m) \ast \phi(m) = q \ast q = p.\]
\[\phi(m \circ k) = \phi(n) = s\) and \(\phi(m) \ast \phi(k) = q \ast p = s.\]
\[\phi(m \circ n) = \phi(l) = r\) and \(\phi(m) \ast \phi(n) = q \ast s = r.\]
\[\phi(n \circ n) = \phi(k) = p\) and \(\phi(n) \ast \phi(n) = s \ast s = p.\]
\[\phi(n \circ k) = \phi(m) = q\) and \(\phi(n) \ast \phi(k) = s \ast p = q.\]
\[\phi(n \circ m) = \phi(l) = r\) and \(\phi(n) \ast \phi(m) = s \ast q = r.\]

\[\therefore \phi\) is an isomorphism.

* Note: the other possible isomorphism is: \(l \to r, k \to p, m \to s, n \to q.\)

Blunders (-3)
B1 Error in selecting isomorphism
B2 Not fully justified
QUESTION 11

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (10, 10) marks</td>
<td>Att (3, 3)</td>
</tr>
</tbody>
</table>

11 (a) Find the coordinates of the point that is invariant under the transformation

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix}.
\]

\[
\begin{aligned}
&x + 3y = -5 \\
&4x - 6y = -2
\end{aligned}
\Rightarrow
\begin{aligned}
&6x = -12 \\
&x = -2
\end{aligned}
\Rightarrow
\begin{aligned}
x & = -2 \\
y & = -1
\end{aligned}

\Rightarrow \text{is an invariant point.}

Blunders (-3)
B1 Error in multiplication or addition of matrices
B2 Error in setting up equations
B3 Error in solving equations

Slips (-1)
S1 Errors in calculations
11 (b) Prove that a similarity transformation maps the circumcentre of a triangle to the circumcentre of the image of the triangle.

(b) Circumcentre 5 marks Att 2
Midpoints 5 marks Att 2
Perpendicularity 5 marks Att 2
Finish 5 marks Att 2

\[\begin{align*}
\text{dg and eg are the perpendicular bisectors of } [ab] \text{ and } [ac] \text{ respectively.} \\
\therefore g \text{ is the circumcentre of } \triangle abc. \\
d \text{ and } e \text{ are the mid-points of } [ab] \text{ and } [ac] \text{ respectively } \Rightarrow d' \text{ and } e' \text{ are the mid-points of } [a'b'] \text{ and } [a'c'] \text{ respectively, as mid-point is an invariant map.} \\
dg \perp ab \text{ and } eg \perp ac \Rightarrow dg' \perp a'b' \text{ and } e'g' \perp a'c' \text{ as } f \text{ is a similarity transformation.} \\
\therefore g' \text{ is the circumcentre of } \triangle a'b'c' \text{ and } f(g) = g'.
\end{align*}\]

Blunders (-3)
B1 Error in finding circumcentre
B2 Fails to identify mid points
B3 Fails to identify perpendiculars
B4 No conclusion
(c) (i) 10 marks

11 (c) (i)

\[ x = ax' \text{ and } y = by'. \]

\[ : \quad f(E) : \frac{a^2 x^2}{a^2} + \frac{b^2 y^2}{b^2} = 1 \Rightarrow x^2 + y^2 = 1. \]

\[ \therefore \]

**Blunders (-3)**

B1  Error in finding images

B2  Error in finding equation of circle

(c) (ii) 10 marks

11 (c) (ii)

By \( f \), \( E \) maps to \( C \) and \( L, K, D \) map onto \( L', K', D' \) respectively.

But \( L' \perp D' \) and \( K' \perp D' \) as tangent to a circle is perpendicular to diameter at point of contact.

\[ \therefore \quad L' \text{ is parallel to } K'. \]

\[ \therefore \quad f^{-1}(L) \text{ is parallel to } f^{-1}(K') \text{ as parallelism is invariant}. \]

\[ \therefore \quad L \text{ is parallel to } K. \]

**Blunders (-3)**

B1  Fails to define tangent to circle

B2  Fails to mention invariance of parallel lines
Ba chóir marcanna de réir an gnáthráta a bhronadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d’iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú síos.

Déantar an cinneadh agus an riomhaireacht faoin marc bónais i gcás gach páipéar ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an gnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. 198 marc × 5% = 9·9 ⇒ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, riomhtar an bónas de réir na foirmle [300 – bunmharc] × 15%, agus an marc bónais sin a shlánú síos. In ionad an riomhaireacht sin a dhéanamh, is féidir úsáid a bhaínt as an tábla thíos.

<table>
<thead>
<tr>
<th>Bunmharc</th>
<th>Marc Bónais</th>
</tr>
</thead>
<tbody>
<tr>
<td>226</td>
<td>11</td>
</tr>
<tr>
<td>227 – 233</td>
<td>10</td>
</tr>
<tr>
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<td>241 – 246</td>
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<td>254 – 260</td>
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