LEAVING CERTIFICATE EXAMINATION, 2006

MATHEMATICS – HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 8 JUNE – MORNING, 9:30 to 12:00

Attempt SIX QUESTIONS (50 marks each).

WARNING: Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.
1. (a) Find the real number $a$ such that for all $x \neq 9$,
\[
\frac{x - 9}{\sqrt{x - 3}} = \sqrt{x} + a.
\]

(b) $f(x) = 3x^3 + mx^2 - 17x + n$, where $m$ and $n$ are constants.
Given that $x - 3$ and $x + 2$ are factors of $f(x)$, find the value of $m$ and the value of $n$.

(c) $x^2 - t$ is a factor of $x^3 - px^2 - qx + r$.

(i) Show that $pq = r$.

(ii) Express the roots of $x^3 - px^2 - qx + r = 0$ in terms of $p$ and $q$.

2. (a) Solve the simultaneous equations
\[
\begin{align*}
y &= 2x - 5 \\
x^2 + xy &= 2.
\end{align*}
\]

(b) (i) Find the range of values of $t \in \mathbb{R}$ for which the quadratic equation
\[(2t - 1)x^2 + 5tx + 2t = 0
\]
has real roots.

(ii) Explain why the roots are real when $t$ is an integer.

(c) $f(x) = 1 - b^{2x}$ and $g(x) = b^{1+2x}$, where $b$ is a positive real number.
Find, in terms of $b$, the value of $x$ for which $f(x) = g(x)$. 

3. (a) Given that \( z = 2 + i \), where \( i^2 = -1 \), find the real number \( d \) such that \( z + \frac{d}{z} \) is real.

(b) (i) Use matrix methods to solve the simultaneous equations
\[
\begin{align*}
4x - 2y & = 5 \\
8x + 3y & = -4
\end{align*}
\]

(ii) Find the two values of \( k \) which satisfy the matrix equation
\[
\begin{pmatrix}
1 & k \\
-2 & 1
\end{pmatrix}
\begin{pmatrix}
3 \\
4
\end{pmatrix}
= 11.
\]

(c) (i) Express \(-8 - 8\sqrt{3} i\) in the form \( r(\cos \theta + is \sin \theta) \).

(ii) Hence find \((-8 - 8\sqrt{3} i)^3\).

(iii) Find the four complex numbers \( z \) such that
\( z^4 = -8 - 8\sqrt{3} i \).
Give your answers in the form \( a + bi \), with \( a \) and \( b \) fully evaluated.

4. (a) \(-2 + 2 + 6 + \ldots + (4n - 6)\) are the first \( n \) terms of an arithmetic series. \( S_n \), the sum of these \( n \) terms, is 160. Find the value of \( n \).

(b) The sum to infinity of a geometric series is \( \frac{9}{2} \).
The second term of the series is \(-2\).
Find the value of \( r \), the common ratio of the series.

(c) The sequence \( u_1, u_2, u_3, \ldots \), defined by \( u_1 = 3 \) and \( u_{n+1} = 2u_n + 3 \), is as follows: 3, 9, 21, 45, 93\ldots

(i) Find \( u_6 \), and verify that it is equal to the sum of the first six terms of a geometric series with first term 3 and common ratio 2.

(ii) Given that, for all \( k \), \( u_k \) is the sum of the first \( k \) terms of a geometric series with first term 3 and common ratio 2, find \( \sum_{k=1}^{n} u_k \).
5. (a) Find the value of the middle term of the binomial expansion of 
\[ \left( \frac{x - y}{y - x} \right)^8. \]

(b) (i) Express \( \frac{2}{(r+1)(r+3)} \) in the form \( \frac{A}{r+1} + \frac{B}{r+3} \).

(ii) Hence find \( \sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} \).

(iii) Hence evaluate \( \sum_{r=1}^{\infty} \frac{2}{(r+1)(r+3)} \).

(c) (i) Given two real numbers \( a \) and \( b \), where \( a > 1 \) and \( b > 1 \), prove that 
\[ \frac{1}{\log_b a} + \frac{1}{\log_a b} \geq 2. \]

(ii) Under what condition is \( \frac{1}{\log_b a} + \frac{1}{\log_a b} = 2. \)

6. (a) Differentiate \( \sqrt{x}(x + 2) \) with respect to \( x \).

(b) The equation of a curve is \( y = 3x^4 - 2x^3 - 9x^2 + 8 \).

(i) Show that the curve has a local maximum at the point (0, 8).

(ii) Find the coordinates of the two local minimum points on the curve.

(iii) Draw a sketch of the curve.

(c) Prove by induction that \( \frac{d}{dx} (x^n) = nx^{n-1}, \ n \geq 1, \ n \in \mathbb{N}. \)
7. (a) Taking $x_1 = 2$ as the first approximation to the real root of the equation $x^3 + x - 9 = 0$, use the Newton-Raphson method to find $x_2$, the second approximation.

(b) The parametric equations of a curve are:

\[ x = 3\cos\theta - \cos^3\theta \]
\[ y = 3\sin\theta - \sin^3\theta, \quad \text{where} \quad 0 < \theta < \frac{\pi}{2}. \]

(i) Find $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$.

(ii) Hence show that $\frac{dy}{dx} = -\frac{1}{\tan^2\theta}$.

(c) Given $y = \ln\left(\frac{3 + x}{\sqrt{9 - x^2}}\right)$, find $\frac{dy}{dx}$ and express it in the form $\frac{a}{b - x^n}$.

8. (a) Find

(i) $\int \sqrt{x} \, dx$

(ii) $\int e^{-2x} \, dx$.

(b) Evaluate

(i) $\int_1^2 x(1 + x^2)^3 \, dx$

(ii) $\int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta$.

(c) The diagram shows the graphs of the curves $y = f(x)$ and $y = g(x)$, where $f(x) = 12 - 3x^2$ and $g(x) = 9x^2$.

(i) Calculate the area of the region enclosed by the curve $y = f(x)$ and the $x$-axis.

(ii) Show that the region enclosed by the curves $y = f(x)$ and $y = g(x)$ has half that area.