Leaving Certificate 2011

Marking Scheme

MATHEMATICS

Higher Level
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>GENERAL GUIDELINES FOR EXAMINERS – PAPER 1</td>
<td>4</td>
</tr>
<tr>
<td>QUESTION 1</td>
<td>5</td>
</tr>
<tr>
<td>QUESTION 2</td>
<td>9</td>
</tr>
<tr>
<td>QUESTION 3</td>
<td>12</td>
</tr>
<tr>
<td>QUESTION 4</td>
<td>15</td>
</tr>
<tr>
<td>QUESTION 5</td>
<td>17</td>
</tr>
<tr>
<td>QUESTION 6</td>
<td>21</td>
</tr>
<tr>
<td>QUESTION 7</td>
<td>24</td>
</tr>
<tr>
<td>QUESTION 8</td>
<td>28</td>
</tr>
<tr>
<td>GENERAL GUIDELINES FOR EXAMINERS – PAPER 2</td>
<td>35</td>
</tr>
<tr>
<td>QUESTION 1</td>
<td>36</td>
</tr>
<tr>
<td>QUESTION 2</td>
<td>41</td>
</tr>
<tr>
<td>QUESTION 3</td>
<td>44</td>
</tr>
<tr>
<td>QUESTION 4</td>
<td>47</td>
</tr>
<tr>
<td>QUESTION 5</td>
<td>50</td>
</tr>
<tr>
<td>QUESTION 6</td>
<td>53</td>
</tr>
<tr>
<td>QUESTION 7</td>
<td>56</td>
</tr>
<tr>
<td>QUESTION 8</td>
<td>60</td>
</tr>
<tr>
<td>QUESTION 9</td>
<td>64</td>
</tr>
<tr>
<td>QUESTION 10</td>
<td>67</td>
</tr>
<tr>
<td>QUESTION 11</td>
<td>70</td>
</tr>
<tr>
<td>MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAELGE</td>
<td>73</td>
</tr>
</tbody>
</table>
GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates’ work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).
   
   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,… etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any correct, relevant step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2, … etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The same error in the same section of a question is penalised once only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5·50 may be written as €5,50.
QUESTION 1

Part (a) 15 (10, 5) marks Att (3, 2)
Part (b)  15 (5, 5, 5) marks Att (2, 2, 2)
Part (c)  20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

1. (a) Simplify fully
\[
\frac{x+1}{x-1} - \frac{x-1}{x+1} - \frac{4}{x^2-1}
\]

Setting up fraction 10 marks Att 3
Fully simplified 5 marks Att 2

\[
\frac{x+1}{x-1} - \frac{x-1}{x+1} - 4 = \frac{(x+1)(x+1) - (x-1)(x-1) - 4}{(x+1)(x-1)} = \frac{x^2 + 2x + 1 - x^2 + 2x - 4}{(x+1)(x-1)}
\]
\[
= \frac{4x - 4}{(x+1)(x-1)} = \frac{4(x - 1)}{(x+1)(x-1)} = \frac{4}{x+1}
\]

Blunders (-3)
B1 Factors once only
B2 Indices
B3 Incorrect cancellation

Part (b) 15 (5, 5, 5) marks Att (2, 2, 2)

1 (b) (i) Prove the factor theorem for polynomials of degree 2.
That is, given that \( f(x) = ax^2 + bx + c \) and \( k \) is a number such that \( f(k) = 0 \), prove that \( (x-k) \) is a factor of \( f(x) \).

(ii) The factor theorem also holds for polynomials of higher degree.
Find the values of \( n \) for which \( (x+k) \) is a factor of the polynomial
\( g(x) = x^n + k^n \), where \( k \neq 0 \).

(b) (i) \( f(x) - f(k) \) factorised 5 marks Att 2
Finish 5 marks Att 2

1 (b) (i)
\[
f(x) = ax^2 + bx + c.
\]
\[
f(k) = ak^2 + bk + c.
\]
\[
\therefore f(x) - f(k) = a(x^2 - k^2) + b(x - k) = a(x + k)(x - k) + b(x - k).
\]
\[
\therefore f(x) - f(k) = (x - k)(ax + ak + b).
\]
\[
\therefore (x - k) \text{ is a factor of } f(x) - f(k).
\]
But \( f(k) = 0 \), \( \Rightarrow (x - k) \) is a factor of \( f(x) \).
Blunders (-3)
B1 Indices
B2 Factors
B3 \( f(k) \neq 0 \)

Slips (-1)
S1 Numerical

(b) (i) Setting up division

<table>
<thead>
<tr>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finish</td>
<td></td>
</tr>
</tbody>
</table>

1 (b) (i)

\[
\begin{align*}
  f(x) &= ax^2 + bx + c \\
  f(k) &= ak^2 + bk + c \\
  f(x) - f(k) &= ax^2 + b - ak^2 - bk \\
  ax + (ak + b) &= x - k \\
  ax^2 + bx - ak^2 - bk &= \frac{ax^2 - akx}{(ak + b)x - ak^2 - bk} \\
  f(k) &= 0, \\
  \Rightarrow f(x) &= (x - k)[ax + (ak + b)]
\end{align*}
\]

Blunders (-3)
B1 Indices

Slips (-1)
S1 Numerical
S2 Not changing sign when subtracting in division

(b) (ii)

<table>
<thead>
<tr>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>

1 (b) (ii)

\[
\begin{align*}
  (x + k) \text{ is a factor of } g(x) & \Rightarrow g(-k) = 0. \\
  \therefore ( -k)^n + k^n &= 0 \Rightarrow (-1)^n k^n + k^n = 0. \\
  \therefore n \text{ is odd} & \Rightarrow n = \{1, 3, 5, 7, 9, \ldots \ldots \}.
\end{align*}
\]

Blunders (-3)
B1 Deduction root from factor
B2 Indices
B3 \((-1)^n\)
B4 Solution set not stated
B5 Only one value \(n\)
1 (c) \((x - a)^2\) is a factor of \(2x^3 - 5ax^2 + 8abx - 36a\), where \(a \neq 0\).

Find the possible values of \(a\) and \(b\).

\[
(x - a)^2 = x^2 - 2ax + a^2.
\]

\[
\begin{array}{c}
2x - a \\
\hline
x^2 - 2ax + a^2 \\
\hline
2x^3 - 5ax^2 + 8abx - 36a \\
\hline
2x^3 - 4ax^2 + 2a^2x \\
\hline
-ax^2 - 2a^2x + 8abx - 36a \\
\hline
-ax^2 + 2a^2x - a^3 \\
\hline
-4a^2x + 8abx - 36a + a^3
\end{array}
\]

\[
\therefore (a^3 - 36a)(a - 2b) = 0.
\]

\[
\therefore -4a^2 + 8ab = 0 \Rightarrow a - 2b = 0 \text{ and } a^2 - 36 = 0, \text{ as } a \neq 0.
\]

\[
\therefore a = \pm 6 \text{ and } b = \pm 3.
\]

\[
\text{ie } a = 6 \text{ and } b = 3 \text{ or } a = -6 \text{ and } b = -3.
\]

**Blunders (-3)**

B1 Expansion of \((x - a)^2\) once only
B2 Indices
B3 Not like to like when equating coefficients
B4 Not two values of 1st variable

**Slips (-1)**

S1 Not changing sign when subtracting

**Attempts**

A1 Any effort at division for 2 marks only
A2 \((x-a)\) as factor.

OR
Other factor 5 marks Att 2
Correct multiplication 5 marks Att 2
Equating coefficients 5 marks Att 2
Values 5 marks Att 2

<table>
<thead>
<tr>
<th>1 (c)</th>
</tr>
</thead>
</table>
| One factor \( (x^2 - 2ax + a^2) \)  
Other factor \( (2x - \frac{36}{a}) \)  
\( (x^2 - 2ax + a^2)(2x - \frac{36}{a}) = 2x^3 - 5ax^2 + 8abx - 36a \)  
\( 2x^3 - 4ax^2 + 2a^2x - \frac{36}{a}x^2 + 72x - 36a = 2x^3 + (-5a)x^2 + 8abx - 36a \)  
\( 2x^3 + (-4a - 36/a)x^2 + (2a^2 + 72)x - 36a = 2x^3 + (-5a)x^2 + (8ab)x - 36a \)  
Equating coefficients  
(i) \( (-4a - 36/a) = (-5a) \)  
\( -4a^2 - 36 = -5a^2 \)  
\( a^2 = 36 \)  
\( a = \pm 6 \)  
(ii) \( (2a^2 + 72) = 8ab \)  
\( 72 + 72 = 8ab \)  
\( \Rightarrow ab = 18 \)  
\( a = \pm 6 \)  
\( \Rightarrow b = \pm 3 \) |

**Blunders (-3)**
B1 Indices
B2 Expansion of \((x - a)^2\) once only
B3 Not like to like when equating coefficients
B4 Not 2 values of 1st variable

**Attempts**
A1 Other factor not linear, Att 2 marks only.
QUESTION 2

Part (a) 15 (10, 5) marks Att (3, 2)
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)
Part (c) 15 (5, 5, 5) marks Att (2, 2, 2)

Part (a) 15 (10, 5) marks Att (3, 2)

2 (a) Solve for \( x \): \(|2x - 1| \leq 3\), where \( x \in \mathbb{R} \).

<table>
<thead>
<tr>
<th>Limits</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
</tbody>
</table>

\[ |2x - 1| \leq 3 \Rightarrow -3 \leq 2x - 1 \leq 3. \]
\[ \therefore -1 \leq x \leq 2. \]

Blunders (-3)
B1 Upper limit
B2 Lower limit
B3 Inequality sign
B4 Indices
B5 Incorrect range
B6 No range

Slips (-1)
S1 Numerical
S2 Not \( \geq \) or \( \leq \)

Attempts
A1 Inequality sign ignored

OR

Quadratic inequality factorised 10 marks Att 3
Range 5 marks Att 2

\[ |2x - 1| \leq 3 \]
\[ (2x - 1)^2 \leq 9 \]
\[ 4x^2 - 4x + 1 \leq 9 \]
\[ 4x^2 - 4x - 8 \leq 0 \]
\[ x^2 - x - 2 \leq 0 \]
\[ (x - 2)(x + 1) = 0 \]
\[ \Rightarrow x = 2 \text{ or } x = -1 \]
\[ f(x) \leq 0 \]
\[ -1 \leq x \leq 2 \]
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

2 (b) \( \alpha \) and \( \frac{1}{\alpha} \) are roots of the quadratic equation \( 3kx^2 - 18tx + (2k + 3) = 0 \), where \( t \) and \( k \) are constants.

(i) Find the value of \( k \).

(ii) If one of the roots is four times the other, find the possible values of \( t \).

\[
\begin{align*}
2 (b) \quad & \quad \alpha \quad \text{and} \quad \frac{1}{\alpha} \quad \text{are roots of the quadratic equation} \quad 3kx^2 - 18tx + (2k + 3) = 0, \\
& \quad \text{where} \quad t \quad \text{and} \quad k \quad \text{are constants}. \\
\end{align*}
\]

2 (b) (i) \[
\alpha \left( \frac{1}{\alpha} \right) = \frac{2k + 3}{3k} \quad \Rightarrow \quad \frac{2k + 3}{3k} = 1 \quad \Rightarrow \quad k = 3.
\]

2 (b) (ii) \[
k = 3 \quad \Rightarrow \quad 9x^2 - 18tx + 9 = 0 \quad \Rightarrow \quad x^2 - 2tx + 1 = 0.
\]
\[
\alpha = \frac{4}{\alpha} \quad \Rightarrow \quad \alpha^2 = 4 \quad \Rightarrow \quad \alpha = \pm 2.
\]
\[
\text{Sum of roots} = \alpha + \frac{1}{\alpha} = 2t \quad \Rightarrow \quad t = \frac{1}{2} \left( \pm \frac{5}{4} \right) = \pm \frac{5}{4}.
\]
2 (c) Let \( f(x) = \frac{1}{x^2 - 6x + a} \), where \( a \) is a constant.

(i) Prove that if \( a = 13 \), then \( f(x) > 0 \) for all \( x \in \mathbb{R} \).

(ii) Prove that if \( a = 13 \), then \( f(x) < 1 \) for all \( x \in \mathbb{R} \).

(iii) Find the range of values of \( a \) such that \( 0 < f(x) < 1 \), for all \( x \in \mathbb{R} \).

<table>
<thead>
<tr>
<th>Part (c) (i)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (c) (ii)</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
<tr>
<td>Part (c) (iii)</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
</tbody>
</table>

\[
1 = \frac{1}{x^2 - 6x + 13} = \frac{1}{(x-3)^2 + 4}.
\]

\((x-3)^2 \geq 0 \) for all \( x \in \mathbb{R} \) \implies (x-3)^2 + 4 > 0.

\[
\therefore \frac{1}{x^2 - 6x + 13} > 0 \implies f(x) > 0 \text{ when } a = 13.
\]

\[
1 = \frac{1}{x^2 - 6x + 13} = \frac{1}{(x-3)^2 + 4}.
\]

\((x-3)^2 \geq 0 \implies (x-3)^2 + 4 > 1.

\[
\therefore \frac{1}{x^2 - 6x + 13} < 1 \implies f(x) < 1 \text{ when } a = 13.
\]

\[
1 = \frac{1}{x^2 - 6x + a} = \frac{1}{x^2 - 6x + 9 + (a-9)} = \frac{1}{(x-3)^2 + (a-9)}
\]

So, to get \( f(x) \) always > 0, we need \( a > 9 \), and

To get \( f(x) \) always less than 1, we need denominator always > 1, so \( a > 10 \).

Combining these two conditions yields the overall condition \( a > 10 \).

---

**Blunders (-3)**

B1 Not \((x-3)^2\)

B2 \([(x-3)^2 + 4] \geq 0\)

B3 \((x-3)^2 \geq 0\)

B4 \([(x-3)^2 + 4] \geq 1\)

B5 Deduction each time from work shown

B6 No deduction each time

B7 Inequality sign
QUESTION 3

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>15 (10, 5) marks</th>
<th>Att (3, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>15 (5, 5, 5) marks</td>
<td>Att (2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

### Part (a) 15 (10, 5) marks Att (3, 2)

3. (a) Express \( \frac{1+2i}{2-i} \) in the form of \( a+bi \), where \( i^2 = -1 \).

#### Multiplication by conjugate 10 marks Att 3

<table>
<thead>
<tr>
<th>Value</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 (a)</td>
<td>( \frac{1+2i}{2-i} = \frac{(1+2i)(2+i)}{(2-i)(2+i)} = \frac{2+5i+2i^2}{4-i^2} = \frac{5i}{5} = i )</td>
<td></td>
</tr>
</tbody>
</table>

---

**Blunders (-3)**

B1 Indices

B2 \( i \)

**Slips (-1)**

S1 Numerical

**Attempts**

A1 Not using correct conjugate

### Part (b) 15 (5, 5, 5) marks Att (2, 2, 2)

3 (b) (i) Find the two complex numbers \( a+bi \) such that

\[(a+bi)^2 = -3 + 4i \]

(ii) Hence solve the equation \( x^2 + x + 1 - i = 0 \).

#### (i) Equations 5 marks Att 2

Finish 5 marks Att 2

#### (ii) Solve 5 marks Att 2

3 (b) (i)

\[
(a + bi)^2 = -3 + 4i \quad \Rightarrow \quad a^2 - b^2 + 2abi = -3 + 4i.
\]

\[\therefore a^2 - b^2 = -3 \quad \text{and} \quad ab = 2.\]

\[
b = \frac{2}{a} \quad \Rightarrow \quad a^2 - \frac{4}{a^2} = -3 \quad \Rightarrow \quad a^4 + 3a^2 - 4 = 0.
\]

\[\therefore (a^2 - 1)(a^2 + 4) = 0 \quad \Rightarrow \quad a^2 - 1 = 0 \quad \text{and} \quad a^2 + 4 \neq 0.
\]

\[\therefore a = \pm 1 \quad \Rightarrow \quad b = \pm 2 \quad \Rightarrow \quad \text{solution is} \pm (1 + 2i).\]
3 (b) (ii)

\[ x^2 + x + (1 - i) = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1 - 4(1 - i)}}{2} = \frac{-1 \pm \sqrt{-3 + 4i}}{2}. \]

\[ \therefore x = \frac{-1 \pm (1 + 2i)}{2} \text{ by part (i).} \]

\[ x = \frac{-1 + 1 + 2i}{2} \text{ or } x = \frac{-1 - 1 - 2i}{2} \Rightarrow x = i \text{ or } x = -1 - i. \]

**Blunders (-3)**

B1 Expansion of \((a + ib)^2\)

B2 Indices

B3 \(i\)

B4 Not like to like

B5 Factors

B6 Quadratic formula

B7 Excess values (not real)

B8 Only one complex number found

B9 Incorrect deduction root from function

**Slips (-1)**

S1 Answers not simplified
Let $A$ and $B$ be $2 \times 2$ matrices, where $A$ has an inverse.

Show that $(A^{-1}BA)^n = A^{-1}B^n A$ for all $n \in \mathbb{N}$.

Let $P = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $M = \begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix}$.

(ii) Evaluate $P^{-1}MP$ and hence $(P^{-1}MP)^n$.

(iii) Hence, or otherwise, show that $M^n = M$, for all $n \in \mathbb{N}$.

Part (c) (i) 5 marks

(c) (ii) 5 marks

(P^{-1}MP)^n$ 5 marks

(c) (iii) 5 marks

(Or by induction)

$(A^{-1}BA)^n = (A^{-1}BA)(A^{-1}BA)(A^{-1}BA) \ldots (A^{-1}BA) = A^{-1}B( AA^{-1})B( AA^{-1}) \ldots ( AA^{-1})B = A^{-1}BIBI \ldots IBA = A^{-1}BBBB \ldots BA = A^{-1}B^n A.$

$(P^{-1}MP)^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

3 (c) (ii)

$P^{-1}MP = \frac{1}{(6-5)} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

3 (c) (iii)

$(P^{-1}MP)^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow P^{-1}M^n P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$M^n = P \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} = M$.

Blunders (-3)

B1 \ $P^i$ once only

B2 \ $P^i, P \neq I$

B3 \ Indices

B4 \ Incorrect order of multiplication

Note: $P^i MP$ must be a diagonal matrix in part (c)(ii) to merit 2$^\text{nd}$ 5 marks; otherwise 0 marks.
QUESTION 4

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 40 (5, 5, 5, 5, 10, 5, 5) marks Att (2, 2, 2, 2, 3, 2, 2)

Part (a) 10 (5, 5) marks Att (2, 2, 2)

4(a) In an arithmetic sequence, the third term is $-3$ and the sixth term is $-15$. Find the first term and the common difference.

\[ T_3, T_6 \]
\[ a \text{ and } d \]

5 marks Att 2

5 marks Att 2

4 (a)

\[
\begin{align*}
    a + 2d &= -3 \\
    a + 5d &= -15 \\
    3d &= -12 \\
    \Rightarrow d &= -4 \text{ and } a = 5.
\end{align*}
\]

First term $= 5$, common difference $= -4$.

(NOTE: $a$ and $d$ can be in any order)

Blunders (-3)
B1 Term of arithmetic sequence
B2 Formula for term once only
B3 Incorrect $a$
B4 Incorrect $d$

Slips (-1)
S1 Numerical

Part (b) 40 (5, 5, 5, 5, 10, 5, 5) marks Att (2, 2, 2, 2, 3, 2, 2)

4 (b) Let $u_n = l \left( \frac{1}{2} \right)^n + m(-1)^n$ for all $n \in \mathbb{N}$.

(i) Verify that $u_n$ satisfies the equation $2u_{n+2} + u_{n+1} - u_n = 0$.

(ii) If $a_k = u_k + u_{k+1}$, express $a_k$ in terms of $k$ and $l$.

(iii) For $l > 0$, find $\sum_{k=1}^{\infty} a_k$, in terms of $l$.

(iv) Find the least positive integer $n$ for which $\sum_{k=1}^{n} a_k > (0.99) \sum_{k=1}^{\infty} a_k$.
(b) (i) Correct \( u_{n+1} \) and \( u_{n+2} \)  
Verify  
5 marks  Att 2  
(b) (ii) Correct \( u_{k+1} \)  
5 marks  Att 2  
Express  
5 marks  Att 2  
(b) (iii) \( S_\infty \)  
10 marks  Att 3  
(b) (iv) \( S_n \)  
5 marks  Att 2  
Least value \( n \)  
5 marks  Att 2

4 (b) (i)  
\[
2u_{n+2} + u_{n+1} - u_n = 2l\left(\frac{1}{2}\right)^{n+2} + 2m(-1)^{n+2} + l\left(\frac{1}{2}\right)^{n+1} + m(-1)^{n+1} - l\left(\frac{1}{2}\right)^n - m(-1)^n.
\]
\[
= l\left(\frac{1}{2}\right)^n \left(\frac{1}{2} + \frac{1}{2} - 1\right) + m(-1)^n(2-1-1) = 0.
\]

4 (b) (ii)  
\[
a_k = u_k + u_{k+1} \quad \Rightarrow \quad a_k = l\left(\frac{1}{2}\right)^k + m(-1)^k + l\left(\frac{1}{2}\right)^{k+1} + m(-1)^{k+1}.
\]
\[
\therefore \quad a_k = l\left(\frac{1}{2}\right)^k \left(\frac{3}{2}\right) + m(-1)^k(1-1)
\]
\[
= \frac{3}{2} l\left(\frac{1}{2}\right)^k.
\]

4 (b) (iii)  
\[
\sum_{k=1}^{\infty} a_k = \frac{3}{2} l\left(\frac{1}{2}\right) + \frac{3}{2} l\left(\frac{1}{2}\right)^2 + \frac{3}{2} l\left(\frac{1}{2}\right)^3 + \cdots + \frac{3}{2} l\left(\frac{1}{2}\right)^k + \cdots
\]

This is an infinite geometric series.  \( \therefore \quad \sum_{k=1}^{\infty} a_k = \frac{\frac{3}{4} l}{1-\frac{1}{2}} = \frac{3}{2} l. \)

4 (b) (iv)  
\[
\sum_{k=1}^{n} a_k = \frac{3}{4} l \left[1 - \left(\frac{1}{2}\right)^n\right] = \frac{3}{2} l \left[1 - \left(\frac{1}{2}\right)^n\right].
\]
\[
\sum_{k=1}^{n} a_k > (0.99) \sum_{k=1}^{n} a_k \quad \Rightarrow \quad \frac{3}{2} l \left[1 - \left(\frac{1}{2}\right)^n\right] > (0.99) \frac{3}{2} l.
\]
\[
\therefore \quad 1 - \left(\frac{1}{2}\right)^n > 0.99 \quad \Rightarrow \quad \left(\frac{1}{2}\right)^n < 0.01 \quad \Rightarrow \quad n = 7.
\]

Blunders (-3)  
B1 In \( u_{n+1} \) once only  
B2 In \( u_{n+2} \) once only  
B3 Indices  
B4 \((-1)^n\)  
B5 Sum of geometric progression to infinity  
B6 Incorrect \( a \)  
B7 Incorrect \( r \)  
B8 Sum of \( n \) terms of geometric progression  
B9 Not using correct values in (iv) once only  
B10 Logs laws  
B11 Not least integer
Part (a) 10 (5, 5) marks  
Att (2, 2)

(a) Find the coefficient of $x^8$ in the expansion of $(x^2 - 1)^{10}$.

$[x^2 + (-1)]^{10}$ Let $u_{r+1}$ be the $r$th term.

$$u_{r+1} = \binom{10}{r} (x^2)^{10-r} (-1)^r$$

$$\Rightarrow k(x^{20-2r}) = k(x^8)$$

$$\Rightarrow 20 - 2r = 8$$

$$12 = 2r$$

$$r = 6$$

Term: $u_7 = \binom{10}{6} (x^2)^{4} (-1)^6 = \binom{10}{4} x^8 = 210x^8$

Coefficient: 210

OR

$$[x^2 + (-1)]^{10} = (x^2)^{10} + \binom{10}{1} (x^2)^{9} (-1)^1 + \binom{10}{2} (x^2)^{8} (-1)^2$$

$$+ \binom{10}{3} (x^2)^{7} (-1)^3 + \binom{10}{4} (x^2)^{6} (-1)^4$$

$$+ \binom{10}{5} (x^2)^{5} (-1)^5 + \binom{10}{6} (x^2)^{4} (-1)^6 + \cdots$$

$$\Rightarrow u_7 = \binom{10}{6} (x^8)(1) = 210x^8$$

Coefficient: 210

Blunders (-3)

B1 General term
B2 Errors in binomial expansion once only
B3 Indices

B4 Error value $\binom{n}{r}$ or no value $\binom{n}{r}$. 
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

5 (b)

(i) Solve the equation:
\[ \log_2 x - \log_2 (x-1) = 4 \log_4 2. \]

(ii) Solve the equation:
\[ 3^{2x+1} - 17(3)^x - 6 = 0. \]

Give your answer correct to two decimal places.

### Part (b) (i) \( \log f(x) = 2 \)

<table>
<thead>
<tr>
<th>Value of ( x )</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>

\[ \log_2 x - \log_2 (x-1) = 4 \log_4 2 \]

\[ \therefore \log_2 \frac{x}{x-1} = \log_4 16 = 2 \]

\[ \therefore \frac{x}{x-1} = 4 \Rightarrow 4x - 4 = x \Rightarrow x = \frac{4}{3}. \]

**Blunders (-3)**
- B1 Logs laws
- B2 Indices

**Worthless**
- W1 Drops ‘log’

### Part (b) (ii) Quadratic factorised

<table>
<thead>
<tr>
<th>Value of ( x )</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>

\[ 3^{2x+1} - 17(3)^x - 6 = 0. \] Let \( y = 3^x \).

\[ \therefore 3y^2 - 17y - 6 = 0. \]

\[ (y-6)(3y+1) = 0 \Rightarrow y = 6, y \neq -\frac{1}{3}. \]

\[ \therefore 3^x = 6 \Rightarrow x \log_e 3 = \log_e 6 \Rightarrow x = \frac{\log_e 6}{\log_e 3} = 1.63. \]

**Blunders (-3)**
- B1 Indices
- B2 Factors once only
- B3 Root formula once only
- B4 Logs
- B5 Uses \( y = -\frac{1}{3} \)

**Slips (-1)**
- S1 Numerical
- S2 Not to 2 decimal places

**Attempts**
- A1 Not quadratic equation
- A2 Correct answer by trial and error
Prove by induction that 9 is a factor of $5^{2n+1} + 2^{4n+2}$, for all $n \in \mathbb{N}$.

Test for $n = 1$.

$P(1): \quad 5^3 + 2^6 = 125 + 64 = 189 = 9 \times 21.$

$\therefore$ True for $n = 1$.

Assume true for $n = k$.

$P(k): \quad 5^{2k+1} + 2^{4k+2} \text{ is divisible by } 9.$

Test for $n = k + 1$.

$P(k + 1): \quad 5^{2k+3} + 2^{4k+6} = 25 \cdot 5^{2k+1} + 16 \cdot 2^{4k+2} = (9 + 16) \cdot 5^{2k+1} + 16 \cdot 2^{4k+2}$

$= 9 \cdot 5^{2k+1} + 16 \left( 5^{2k+1} + 2^{4k+2} \right), \text{ which is divisible by } 9.$

$\therefore$ True for $n = k + 1$.

So, whenever $P(k)$ is true, $P(k+1)$ true.

Since $P(1)$ true, then, by induction, $P(n)$ true for all $n \in \mathbb{N}$.

* Note: accept $n = 0$ as base case.

OR

To prove $5^{2n+1} + 2^{4n+2}$ is divisible by 9.

Test $n=1$

$P(1): \quad 5^3 + 2^6 = 125 + 64 = 189 = 9(21)$

$\Rightarrow$ True for $n = 1$

Assume true for $n = k$

$P(k): \quad (5^{2k+1} + 2^{4k+2}) \text{ is divisible by } 9 \quad (*)$

To prove: $(5^{2k+3} + 2^{4k+6}) \text{ is divisible by } 9$

Let $f(k) = 5^{2k+1} + 2^{4k+2}$

Given the assumption that $f(k)$ is divisible by 9, then $f(k+1)$ will be divisible by 9 if and only if $\lfloor f(k+1) - f(k) \rfloor$ is divisible by 9.

...ctd.
\[
\begin{align*}
  f(k+1) - f(k) &= (5^{2k+3} + 2^{4k+6}) - (5^{2k+1} + 2^{4k+2}) \\
  &= 25(5^{2k+1}) + 16(2^{4k+2}) - 5^{2k+1} - 2^{4k+2} \\
  &= 24(5^{2k+1}) + 15(2^{4k+2}) \\
  &= (27 - 3)(5^{2k+1}) + (18 - 3)(2^{4k+2}) \\
  &= 27(5^{2k+1}) + 18(2^{4k+2}) - 3(5^{2k+1}) - 3(2^{4k+2}) \\
  &= 9[3(5^{2k+1}) + 2(2^{4k+2})] - 3[5^{2k+1} + 2^{4k+2}] \\
  \downarrow &\quad \downarrow \\
  \text{Divisible by 9} &\quad \text{Divisible by 9 from (*) above}
\end{align*}
\]

\[\Rightarrow f(k+1) - f(k) \text{ is divisible by 9}\]

So whenever \(P(k)\) true, \(P(k+1)\) is true. Since \(P(1)\) is true, then by induction \(P(2), P(3), P(4)\) …… are all true.

\textit{Blunders (-3)}
B1 Indices
B2 \(n \geq 2\)

\textit{Slips (-1)}
S1 Numerical

Note: Must prove \(P(1)\) step. Not sufficient to state \(P(n)\) true for \(n=1\)
QUESTION 6

Part (a) 15 marks Att 5

(a) Differentiate $\cos^2 x$ with respect to $x$.

\[ f(x) = \cos^2 x \Rightarrow f'(x) = -2\cos x \sin x. \]

Blunders (-3)
B1 Differentiation

Attempts
A1 Error in differentiation formula (chain rule)

Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

(b) The equation of a curve is $y = e^{-x^2}$.

(i) Find $\frac{dy}{dx}$.

(ii) Find the co-ordinates of the turning point of the curve.

(iii) Determine whether this turning point is a local maximum or a local minimum.

Part (b) (i) 5 marks Att 2

(ii) $f''(x) = 0$

Turning point 5 marks Att 2

(iii) 5 marks Att 2

6 (b) (i)

\[ \frac{dy}{dx} = e^{-x^2}(-2x). \]

6 (b) (ii)

\[ \frac{dy}{dx} = 0 \Rightarrow e^{-x^2}(-2x) = 0 \Rightarrow x = 0 \text{ and } y = 1. \text{ Turning point is } (0, 1). \]

6 (b) (iii)

\[ \frac{d^2y}{dx^2} = e^{-x^2}(-2x)(-2x) - 2e^{-x^2} = e^{-x^2}(4x^2 - 2). \]

For $x = 0$, $\frac{d^2y}{dx^2} = -2e^0 = -2 < 0 \Rightarrow (0, 1) \text{ is a local maximum.}$
Blunders (-3)
B1 Indices
B2 Differentiation
B3 \( e^{-x^2} = 0 \)
B4 No 2\(^{nd}\) differential

Attempts
A1 Error in differentiation formula (chain rule)

Note: Over simplified work in (i) can lead to attempt at most in (ii) and (iii).

Part (c) 15 (5, 5, 5) marks Att (2, 2, 2)

6 (c) The function \( f \) is defined as \( x \to \frac{2x}{x+1} \), where \( x \in \mathbb{R}\backslash\{-1\} \).

(i) Find the equations of the asymptotes of the curve \( y = f(x) \).
(ii) \( P \) and \( Q \) are distinct points on the curve \( y = f(x) \). The tangent at \( Q \) is parallel to the tangent at \( P \). The co-ordinates of \( P \) are \((1,1)\).

Find the co-ordinates of \( Q \).
(iii) Verify that the point of intersection of the asymptotes is the midpoint of \([PQ]\).

Part (c) (i) 5 marks Att 2
(ii) 5 marks Att 2
(iii) 5 marks Att 2

6 (c) (i)
\[ x = -1 \] is the vertical asymptote.
\[ \lim_{x \to -1} \frac{2x}{x+1} = \lim_{x \to -1} \frac{2}{1} = 2 \Rightarrow y = 2 \] is a horizontal asymptote.

6 (c) (ii)
\[ f'(x) = \frac{2(x+1) - 2x(1)}{(x+1)^2} = \frac{2}{(x+1)^2} \]. Slope at \( P(1,1) = \frac{2}{4} = \frac{1}{2} \).
Slope at \( Q = \frac{1}{2} \quad \Rightarrow \quad \frac{2}{(x+1)^2} = \frac{1}{2} \quad \Rightarrow \quad (x+1)^2 = 4. \]
\[ \therefore \ x+1 = \pm 2 \quad \Rightarrow \quad x = 1 \ or \ x = -3. \quad \therefore \ Q = (-3,3). \]

OR
\((x + 1)^2 = 4\)
\[x^2 + 2x + 1 - 4 = 0\]
\[x^2 + 2x - 3 = 0\]
\[(x + 3)(x - 1) = 0\]
\[\Rightarrow x + 3 = 0 \quad \text{or} \quad x - 1 = 0\]
\[x = -3 \quad \text{or} \quad x = 1\]
\[\downarrow \quad \downarrow\]
\[Q(-3,3) \quad P(1,1)\]

6 (c) (iii) Asymptotes intersect at \((-1, 2)\),
\[P(1,1) \text{ and } Q(-3, 3).\]
Mid-point of \([PQ]\) is \((-1, 2)\).

**Blunders (-3)**
B1 Asymptotes
B2 Limits
B3 Differentiation
B4 Indices
B5 Formula for mid-point line

**Slips (-1)**
S1 Numerical

**Attempts**
A1 Error in differentiation formula

Note: Cannot get 2nd 5 marks in (c) (ii) if slope at \(Q\) not equal to slope at \(P\).


QUESTION 7

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 25 (10, 10, 5) marks Att (3, 3, 2)
Part (c) 15 (10, 5) marks Att (3, 2)

Part (a) 10 (5, 5) marks Att (2, 2)

7 (a) Find the slope of the tangent to the curve \( x^2 + y^3 = x - 2 \) at the point (3, −2).

Differentiation 5 marks Att 2
Slope 5 marks Att 2

7 (a) 
\[
2x + 3y^2 \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1 - 2x}{3y^2}. \quad \therefore \text{Slope of tangent at } (3, -2) = \frac{-5}{12}.
\]

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Incorrect value of \( x \) or no value of \( x \) in slope
B4 Incorrect value of \( y \) or no value of \( y \) in slope

Slips (-1)
S1 Numerical

Attempts
A1 Error in differentiation formula
A2 \( \frac{dy}{dx} = 2x + 3y^2 \frac{dy}{dx} = 1 \) and uses the two \( \left( \frac{dy}{dx} \right) \) terms
A curve is defined by the parametric equations
\[ x = \frac{t-1}{t+1} \quad \text{and} \quad y = \frac{-4t}{(t+1)^2}, \] where \( t \neq -1 \).

(i) Find \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \).

(ii) Hence find \( \frac{dy}{dx} \), and express your answer in terms of \( x \).

\[
\frac{dx}{dt} = \frac{(t+1)(-1)-(t-1)(1)}{(t+1)^2} = \frac{2}{(t+1)^2},
\frac{dy}{dt} = \frac{-4(t+1)^2 + 4t(2)(t+1)}{(t+1)^4} = \frac{-4(t+1) + 8t}{(t+1)^3} = \frac{4(t-1)}{(t+1)^3}.
\]

\[
\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{4(t-1)}{(t+1)^3} \times \frac{(t+1)^2}{2} = \frac{2(t-1)}{t+1} = 2x.
\]

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Error in getting \( \frac{dy}{dx} \)

Attempts
A1 Error in differentiation formula
The functions $f$ and $g$ are defined on the domain $\mathbb{R}\{−1, 0\}$ as follows:

$$f : x \to \tan^{-1}\left(\frac{-x}{x+1}\right) \quad \text{and} \quad g : x \to \tan^{-1}\left(\frac{x+1}{x}\right).$$

(i) Show that $f'(x) = \frac{-1}{2x^2 + 2x + 1}$.

(ii) It can be shown that $f'(x) = g'(x)$.

One of the three diagrams A, B, or C below represents parts of the graphs of $f$ and $g$. Based only on the derivatives, state which diagram is the correct one, and state also why each of the other two diagrams is incorrect.

**Diagram A**

- $f(x)$
- $g(x)$

**Diagram B**

- $f(x)$
- $g(x)$

**Diagram C**

- $f(x)$
- $g(x)$

---

**c(i)** 10 marks

7 (c) (i)

$$f(x) : x \to \tan^{-1}\left(\frac{-x}{x+1}\right)$$

$$f'(x) = \frac{1}{1+\left(\frac{-x}{x+1}\right)^2} \times \frac{-1(x+1)+x(1)}{(x+1)^2} = \frac{(x+1)^2}{x^2 + 2x + 1 + x^2} \times \frac{-1}{(x+1)^2} = \frac{-1}{2x^2 + 2x + 1}.$$
\[
\frac{dy}{dx} = -\cos^2 y \\
= \frac{-1}{(x+1)^2} \cdot \frac{(x+1)^2}{2x^2 + 2x + 1} \\
= \frac{-1}{2x^2 + 2x + 1}
\]

**Blunders (-3)**
B1 Differentiation
B2 Indices
B3 Error in value of tan \( y \)
B4 Error in value of cos \( y \)
B5 Sides of triangle once only

**Attempts**
A1 Error in differentiation formula and hence Att 2 at most in simplification

<table>
<thead>
<tr>
<th>Part (c) (ii)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 (c) (ii)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Diagram A is correct. 
It cannot be Diagram B, as these curves are not “parallel” (i.e. identical up to a vertical shift, which is necessary because their derivatives are equal for all \( x \)). 
It cannot be Diagram C as these graphs are increasing, whereas they should be decreasing, because their derivatives are negative for \( x > 0 \). |

**OR**

Given \( f'(x) = g'(x) \)
\( \Rightarrow m_1 = m_2 \) (same slopes)
\( \Rightarrow \) parallel curves
\( f'(x) = \frac{-1}{2x^2 + 2x + 1} < 0 \) when \( x > 0 \)
\( \Rightarrow \) Both \( f(x) \) and \( g(x) \) are decreasing functions.

Diagram A: correct
Diagram B: not parallel curves
Diagram C: increasing curves

**Blunders (-3)**
B1 Incorrect statement
### QUESTION 8

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>15 marks</th>
<th>Att 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>25 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>10 (5, 5) marks</td>
<td>Att (2, 2)</td>
</tr>
</tbody>
</table>

#### Part (a) 15 marks Att 5

8 (a) Find \( \int \left( x^3 + \sqrt{x} \right) \, dx \).

\[
\int \left( x^3 + \sqrt{x} \right) \, dx = \frac{1}{4}x^4 + \frac{2}{3}x^{3/2} + c.
\]

**Blunders (-3)**
- B1 Integration
- B2 Indices
- B3 No ‘c’

#### Part (b) 25 (5, 5, 5, 5, 5) marks Att (2, 2, 2, 2, 2)

**8 (b) (i)** Evaluate \( \int_{0}^{2} \frac{x + 1}{x^2 + 2x + 2} \, dx \).

**8 (b) (ii)** Evaluate \( \int_{0}^{2} \frac{x^2 + 2x + 2}{x + 1} \, dx \).

**Part (b) (i) Correct substitution 5 marks Att 2**
- Integration 5 marks Att 2
- Finish 5 marks Att 2

8 (b) (i)

\[
\int_{0}^{2} \frac{x + 1}{x^2 + 2x + 2} \, dx = \int_{0}^{2} \frac{1}{2} \left( \frac{1}{x + 1} + \frac{2}{x^2 + 2x + 2} \right) \, dx.
\]

Let \( u = x^2 + 2x + 2 \) \( \Rightarrow \) \( du = (2x + 2)\, dx \).

\[
= \frac{1}{2} \left[ \log_e u \right]_{0}^{2} = \frac{1}{2} \left[ \log_e 10 - \log_e 2 \right] = \frac{1}{2} \log_e 10 \approx 1.50.
\]

**Blunders (-3)**
- B1 Integration
- B2 Differentiation
- B3 Logs
- B4 Limits
- B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits

Slips (-1)
S1 Numerical

### Part (b) (ii) Integration

<table>
<thead>
<tr>
<th></th>
<th>Integration</th>
<th>5 marks</th>
<th>Finish</th>
<th>5 marks</th>
<th>Att 2</th>
<th></th>
</tr>
</thead>
</table>
| 8 (b) (ii) | \[ \int_0^2 \frac{x^2 + 2x + 2}{x + 1} \, dx = \int_0^2 \frac{(x + 1)^2 + 1}{x + 1} \, dx = \int_0^2 \left( (x + 1) + \frac{1}{x + 1} \right) \, dx \]

\[ = \left[ \frac{1}{2} x^2 + x + \log_e (x + 1) \right]_0^2 = 2 + 2 + \log_e 3 = 4 + \log_e 3. \]

OR

\[ \int \frac{x^2 + 2x + 2}{x + 1} \, dx \]

\[ = \int \left[ (x + 1) + \frac{1}{x + 1} \right] \, dx \]

\[ = \left[ x + 1 \right] \frac{x + 1}{x^2 + 2x + 2} \]

\[ = \frac{x^2 + x}{x + 2} \]

Finish as above

Blunders (-3)
B1 Integration
B2 Differentiation
B3 Logs
B4 Limits
B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits

Slips (-1)
S1 Numerical
S2 Not changing sign when subtracting in division
Use integration methods to establish the formula \( A = \pi r^2 \) for the area of a disc of radius \( r \).

\[ x^2 + y^2 = r^2 \] is a circle, centre \((0, 0)\), radius \( r \).

Area of disc \( A = 4 \int_0^r \sqrt{r^2 - x^2} \, dx \)

Let \( x = r \sin \theta \Rightarrow dx = r \cos \theta \, d\theta \).

\[ \therefore A = 4 \int_0^\frac{\pi}{2} \sqrt{r^2 - r^2 \sin^2 \theta} \, r \cos \theta \, d\theta = 4 \int_0^\frac{\pi}{2} \sqrt{r^2(1 - \sin^2 \theta)} \, r \cos \theta \, d\theta \]

\[ = 4 \int_0^\frac{\pi}{2} r^2 \cos^2 \theta \, d\theta = (4r^2) \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^\frac{\pi}{2} \]

\[ = 2r^2 \left[ \left( \frac{\pi}{2} + \sin \pi \right) - (0 + 0) \right] \]

\[ \therefore A = 2r^2 \left( \frac{\pi}{2} \right) = \pi r^2. \]

OR

\[ \frac{x}{r} = \sin \theta \Rightarrow x = r \sin \theta \]

\[ \frac{dx}{d\theta} = r \cos \theta \Rightarrow dx = r \cos \theta \, d\theta \]

From diagram: \( \cos \theta = \frac{\sqrt{r^2 - x^2}}{r} \Rightarrow r \cos \theta = \sqrt{r^2 - x^2} \)

\[ A = 4 \int_0^r \sqrt{r^2 - x^2} \, dx \]

\[ = 4 \int (r \cos \theta) \cdot (r \cos \theta) \, d\theta \]

\[ = 4 \int r^2 \cos^2 \theta \, d\theta \quad \text{etc.} \]

Blunders (-3)
B1 Integration
B2 Differentiation
B3 Trig formula
B4 Indices
B5 Limits
B6 Incorrect order in applying limits
B7  Not calculating substituted limits  
B8  Not changing limits  
B9  Definition of $\sin \theta$  
B10 Definition of $\cos \theta$

Slips (-1)  
S1  Numerical  
S2  Trig value or no trig value

Attempts  
A1  Error in differentiation formula or rules of integration

Worthless  
W1  $x = r \sin \theta$ or $x = r \cos \theta$ not used in integration: 0 marks for 2nd 5
GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates’ work as follows:
   • Blunders - mathematical errors/omissions (-3)
   • Slips - numerical errors (-1)
   • Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   • any correct, relevant step in a part of a question merits at least the attempt mark for that part
   • if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   • a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2, …etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The same error in the same section of a question is penalised once only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5·50 may be written as €5,50.
QUESTION 1

Part (a) 10 marks Att 3

Part (b) 25 (10, 5, 5, 5) marks Att (3, 2, 2, 2)

Part (c) 15 (10, 5) marks Att (3, 2)

1 (a) The following parametric equations define a circle:

\[ x = 2 + 3 \sin \theta, \quad y = 3 \cos \theta, \text{ where } \theta \in \mathbb{R}. \]

What is the Cartesian equation of the circle?

\[ (x - 2)^2 + y^2 = 9. \]

\[ \therefore (x - 2)^2 + y^2 = 9. \]

OR

\[ x^2 = 4 + 12 \sin \theta + 9 \sin^2 \theta \quad \text{and} \quad y^2 = 9 \cos^2 \theta \]

\[ \Rightarrow x^2 + y^2 = 4 + 12 \sin \theta + 9 (\sin^2 \theta + \cos^2 \theta) = 13 + 12 \sin \theta \]

\[ \Rightarrow x^2 + y^2 = 13 + 4x - 8 \]

\[ \Rightarrow x^2 + y^2 - 4x - 5 = 0 \]

OR

\[ \cos^2 \theta + \sin^2 \theta = 1 \]

\[ \left( \frac{x - 2}{3} \right)^2 + \left( \frac{y}{3} \right)^2 = 1 \]

\[ \Rightarrow (x - 2)^2 + y^2 = 9 \]

Centre (2,0) and Radius 3 \( \Rightarrow (x - 2)^2 + y^2 = 9 \)

Blunders (-3)

B1 Incorrect squaring (apply once if same type of error)

B2 \( \cos^2 \theta + \sin^2 \theta \neq 1 \)

B3 Incorrect centre or radius

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Effort at expressing \( x^2 \) or \( y^2 \) in terms of \( \theta \)

A2 \( \theta \) not eliminated

A3 Centre (2,0) and/or radius 3 and stops

A4 \( x^2 + y^2 = 9 \) with work

Worthless

W1 \( x^2 + y^2 = 1 \)
Part (b)  25 (10, 5, 5, 5) marks  Att (3, 2, 2, 2)

1 (b) Find the equation of the circle that passes through the points (0, 3), (2, 1) and (6, 5).

(b) One mediator  10 marks  Att 3
2nd mediator  5 marks  Att 2
Centre  5 marks  Att 2
Finish  5 marks  Att 2

1 (b)
Mid-point \([AB] = E(1, 2)\).
Slope of \(AB = \frac{3-1}{0-2} = -1 \Rightarrow \) slope \(EQ = 1\).
\[\therefore \text{Equation } EQ : y - 2 = l(x - 1) \Rightarrow EQ : x - y = -1.\]

Mid-point \([BC] = D(4, 3)\).
Slope of \(BC = \frac{5-1}{6-2} = 1 \Rightarrow \) slope of \(DQ = -1.\)
\[\therefore \text{Equation } DQ : y - 3 = l(x - 4) \Rightarrow DQ : x + y = 7.\]
x - y = -1
x + y = 7
\[2x = 6 \Rightarrow x = 3 \text{ and } y = 4. \therefore \text{Centre } Q \text{ is } (3, 4).\]
\[|AQ| = r = \sqrt{(3-0)^2 + (4-3)^2} = \sqrt{10}. \text{ Equation of circle: } (x - 3)^2 + (y - 4)^2 = 10.\]

OR

(b) An equation in two variables  10 marks  Att 3
Second equation in two variables  5 marks  Att 2
Two values  5 marks  Att 2
Finish  5 marks  Att 2

1(b)
\[x^2 + y^2 + 2gx + 2fy + c = 0\]
\[\Rightarrow 0 + 9 + 2g(0) + 2f(3) + c = 0 \Rightarrow 6f + c = -9.\]
Also \[4 + 1 + 4g + 2f + c = 0 \Rightarrow 4g + 2f + c = -5 \ldots \ldots \ldots \ldots (i)\]
and \[36 + 25 + 12g + 10f + c = 0 \Rightarrow 12g + 10f + c = -61 \ldots \ldots \ldots \ldots (ii)\]
Solving between (i) and (ii) \(g = -3 \text{ and } f = -4\)
\[\Rightarrow 6(-4) + c = -9 \Rightarrow c = 15\]
Equation of circle: \(x^2 + y^2 - 6x - 8y + 15 = 0\)

OR
(b) Appropriate slopes 10 marks  
Establishing semi circle 5 marks  
Centre or radius 5 marks  
Finish 5 marks  

1(b)  
Slope (0,3) and (2,1) \[
\frac{1-3}{2-0} = -1
\]
Slope (2,1) and (6,5) \[
\frac{5-1}{6-2} = 1
\]
\[\Rightarrow\] perpendicular lines.
But angle in a semi-circle right angle \(\Rightarrow\) (0,3) and (6,5) diameter extremities.
Centre of circle (3,4)
Radius: \(\sqrt{(3-0)^2 + (4-3)^2} = \sqrt{10}\)
Equation: \((x - 3)^2 + (y - 4)^2 = 10\)

Blunders (-3)
B1 Incorrect perpendicular slope  
B2 Error in slope formula  
B3 Error in equation of line formula  
B4 Error in radius formula  
B5 Equation of circle incomplete  
B6 Incorrect diameter  
B7 Error in general equation of circle  
B8 Equation of circle but radius not calculated

Slips (-1)
S1 Arithmetic errors

Attempts (3, 2, 2, 2 marks)
A1 Product of perpendicular slopes = -1  
A2 Mixing \(x\) and \(y\) ordinates  
A3 Correct formula with some correct substitution  
A4 Some correct substitution into general equation of circle

Part (c) 15 (10, 5) marks  

1 (c) The circle \(c_1: x^2 + y^2 - 8x + 2y - 23 = 0\) has centre \(A\) and radius \(r_1\).
The circle \(c_2: x^2 + y^2 + 6x + 4y + 3 = 0\) has centre \(B\) and radius \(r_2\).

(i) Show that \(c_1\) and \(c_2\) intersect at two points.
(ii) Show that the tangents to \(c_1\) at these points of intersection pass through the centre of \(c_2\).
Part (c)(i) 10 marks

1 (c) (i)

\[ A(4, -1) \text{ and } r_1 = \sqrt{16+1+23} = \sqrt{40} = 2\sqrt{10}. \]
\[ B(-3, -2) \text{ and } r_2 = \sqrt{9+4-3} = \sqrt{10}. \]
\[ |AB| = \sqrt{(4+3)^2 + (-1+2)^2} = \sqrt{50} = 5\sqrt{2}. \]

So, \( r_1 + r_2 = 3\sqrt{10} = \sqrt{90} > \sqrt{50} \text{ and } |r_1 - r_2| = \sqrt{10} < \sqrt{50} \]

\( \Rightarrow \) circles intersect at two points.

OR

Part (c)(i) 10 marks

1(c)(i)

\[ x^2 + y^2 - 8x + 2y - 23 = 0 \]
\[ x^2 + y^2 + 6x + 4y + 3 = 0 \]
\[ -14x - 2y - 26 = 0 \Rightarrow y = -7x - 13 \]
\[ x^2 + (-7x - 13)^2 - 8x + 2(-7x - 13) - 23 = 0 \]
\[ \Rightarrow 5x^2 + 16x + 12 = 0 \]
\[ \Rightarrow (5x + 6)(x + 2) = 0 \]
\[ \Rightarrow x = -\frac{6}{5}, x = -2 \]
\[ \Rightarrow y = -\frac{23}{5}, y = 1 \]

Two points of intersection \( \left(-\frac{6}{5}, -\frac{23}{5}\right) \) and (-2,1)

Blunders (-3)

B1 Relationship between \( 3\sqrt{10} \) and \( \sqrt{50} \) or \( \sqrt{40} + \sqrt{10} > \sqrt{50} \) not clearly established
B2 Error in squaring
B3 Error in factors
B4 Incorrect conclusion stated or implied

Slips (-1)

S1 Arithmetic errors
S2 Not establishing both cases

Attempts (3 marks)

A1 One centre and radius found
A2 Expressing \( y \) in terms of \( x \) and stops

[39]
Part (c) (ii)  
Let $P$ and $Q$ be the points of intersection of the circles. The tangent to $c_1$ passes through $B$, if and only if $APB$ and $AQB$ are right-angled triangles.

\[ |AP|^2 + |BP|^2 = r_1^2 + r_2^2 = 40 + 10 = 50 = |AB|^2. \]

\[ \therefore \angle APB = 90^\circ \Rightarrow AP \perp PB. \]

\[ \therefore PB \text{ is a tangent to } c_1 \text{ and contains centre } B \text{ of } c_2. \]

Similarly $QB$ is a tangent to $c_1$ and contains centre $B$ of $c_2$.

OR

Part (c)(ii)  
Slope diameter: centre(4,-1) and point of contact (-2,1)
\[ \frac{-1-1}{4+2} = -\frac{1}{3} \Rightarrow \text{slope of tangent equals 3} \]

Equation of tangent: \[ y-1=3(x+2) \Rightarrow 3x-y+7=0 \]
But (-3,-2) lies on tangent since $3(-3)-1(-2)+7 = -9+2+7=0$

Slope (4,-1) and \[ \left(-\frac{6}{5}, -\frac{23}{5}\right) \text{ equals } \frac{9}{13} \Rightarrow \text{slope of tangent equals } \frac{-13}{9} \]

Equation of tangent: \[ y + \frac{23}{5} = \frac{-13}{9} \left(x + \frac{6}{5}\right) \]
But (-3,-2) lies on this tangent since
LHS: $-2 + \frac{23}{5} = \frac{13}{5}$ and RHS: \[ \frac{-13}{9} \left(-\frac{3}{5} + \frac{6}{5}\right) = \frac{-13}{9} \left(-\frac{9}{5}\right) = \frac{13}{5} \]

Blunders (-3)  
B1 Incorrect use of Pythagoras  
B2 One case only  
B3 Incorrect slope or equation of line formula with substitution  
B4 Not verifying centre on tangents

Slips (-1)  
S1 Arithmetic errors

Attempts (2 marks)  
A1 Squaring one radius and stops  
A2 Equation of one tangent only and stops

Misreading(-1)  
M1 Centres interchanged

[40]
QUESTION 2

Part (a) 15 marks Att 5

2 (a) Find the value of \( s \) and the value of \( t \) that satisfy the equation
\[
s(i - 4j) + t(2i + 3j) = 4i - 27j.
\]

2 (a)  15 marks  Att 5

\[
s(i - 4j) + t(2i + 3j) = 4i - 27j
\]
\[
\Rightarrow 4s + 8t = 16
\]
\[
-4s + 3t = -27
\]
\[
11t = -11 \Rightarrow t = -1 \text{ and } s = 6.
\]

Blunders (-3)
B1 One value only

Slips (-1)
S1 Arithmetic errors

Attempts (5 marks)
A1 One equation in \( s \) and \( t \)

Part (b) 20 (10, 10) marks Att (3, 3)

2 (b) \( \overrightarrow{OP} = 3i - 4j \) and \( \overrightarrow{OQ} = 5\left(\overrightarrow{OP}^\perp\right) \).

(i) Find \( \overrightarrow{OQ} \) in terms of \( i \) and \( j \).

(ii) Find \( \cos \angle OQP \), in surd form.

2 (b) (i) 10 marks Att 3

\[
\overrightarrow{OP} = 3i - 4j \Rightarrow \overrightarrow{OP}^\perp = 4i + 3j.
\]
\[
\Rightarrow \overrightarrow{OQ} = 20i + 15j.
\]

Blunders (-3)
B1 Error in \( \overrightarrow{OP}^\perp \)
B2 \( \overrightarrow{OQ} = \left(\overrightarrow{OP}^\perp\right) \)

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 Relationship between a vector and related perpendicular stated or implied
Part (b) (ii)  

$$\cos \angle OQP = \frac{(\overrightarrow{OQ} \cdot \overrightarrow{PQ})}{\|\overrightarrow{OQ}\|\|\overrightarrow{PQ}\|} = \frac{(20\hat{i} + 15\hat{j})(17\hat{i} + 19\hat{j})}{\sqrt{400 + 225} \sqrt{289 + 461}} = \frac{340 + 285}{\sqrt{625 \times 650}} = \frac{625}{5\sqrt{26}} = \frac{5}{\sqrt{26}}.$$ 

Blunders (-3) 
B1 $\overrightarrow{PQ} \neq \overrightarrow{q} - \overrightarrow{p}$ 
B2 Error in modulus formula 
B3 Answer not in single surd 

Slips (-1) 
S1 Arithmetic errors. 

Attempts (3 marks) 
A1 $\cos \angle POQ$ calculated 
A2 $\cos \theta$ formula with some correct substitution 

Part (c) 15 (5, 5, 5) marks 

2 (c) $ABC$ is a triangle and $D$ is the mid-point of $[BC]$. 

(i) Express $\overrightarrow{AB}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$ and express $\overrightarrow{AC}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$. 

(ii) Hence, prove that $|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2|\overrightarrow{AD}|^2 + \frac{1}{2}|\overrightarrow{BC}|^2$. 

Part (c) (i) 10 (5, 5) marks 

2 (c) (i) 

$$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AD} - \frac{1}{2} \overrightarrow{BC}.$$ 

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AD} + \frac{1}{2} \overrightarrow{BC}.$$ 

Blunders (-3) 
B1 $\overrightarrow{DB} \neq -\frac{1}{2} \overrightarrow{BC}$ 
B2 $\overrightarrow{DC} \neq \frac{1}{2} \overrightarrow{BC}$ 

Attempts (2, 2 marks) 
A1 $\overrightarrow{AB}$ and/or $\overrightarrow{AC}$ as the sum of two vectors
Part (c) (ii) 5 marks

\[ |AB|^2 = \overrightarrow{AB} \cdot \overrightarrow{AB} = (\overrightarrow{AD} - \frac{1}{3} \overrightarrow{BC})(\overrightarrow{AD} - \frac{1}{3} \overrightarrow{BC}) = |\overrightarrow{AD}|^2 + \frac{1}{9}|\overrightarrow{BC}|^2 - \frac{1}{3} \overrightarrow{AD} \cdot \overrightarrow{BC} - \frac{1}{3} \overrightarrow{BC} \cdot \overrightarrow{AD} \]

\[ |AC|^2 = \overrightarrow{AC} \cdot \overrightarrow{AC} = (\overrightarrow{AD} + \frac{1}{3} \overrightarrow{BC})(\overrightarrow{AD} + \frac{1}{3} \overrightarrow{BC}) = |\overrightarrow{AD}|^2 + \frac{1}{9}|\overrightarrow{BC}|^2 + \frac{1}{3} \overrightarrow{AD} \cdot \overrightarrow{BC} + \frac{1}{3} \overrightarrow{BC} \cdot \overrightarrow{AD} \]

\[ \therefore |AB|^2 + |AC|^2 = 2|\overrightarrow{AD}|^2 + \frac{1}{3}|\overrightarrow{BC}|^2. \]

Blunders (-3)
B1 Incorrect conclusion or no conclusion implied

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 \( (\overrightarrow{AD} - \frac{1}{3} \overrightarrow{BC})(\overrightarrow{AD} - \frac{1}{3} \overrightarrow{BC}) = |\overrightarrow{AD}|^2 + \frac{1}{9}|\overrightarrow{BC}|^2 \)
A2 \( |AB|^2 \) or \( (\overrightarrow{AD} - \frac{1}{3} \overrightarrow{BC})(\overrightarrow{AD} - \frac{1}{3} \overrightarrow{BC}) = |\overrightarrow{AD}|^2 + \frac{1}{9}|\overrightarrow{BC}|^2 - \overrightarrow{AD} \cdot \overrightarrow{BC} \)
A3 \( \overrightarrow{AB} \cdot \overrightarrow{AB} = |\overrightarrow{AB}|^2 \) or \( |AB|^2 \)

Worthless (0 marks)
W1 \( |AB|^2 = |\overrightarrow{AD}|^2 + \frac{1}{3}|\overrightarrow{BC}|^2 \)
**QUESTION 3**

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>15 marks</th>
<th>Att 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>35 (20, 5, 5, 5) marks</td>
<td>Att (7, 2, 2, 2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>15 marks</th>
<th>Att 5</th>
</tr>
</thead>
</table>

3 (a) \( P \) and \( Q \) are the points \((-1, 4)\) and \((3, 7)\) respectively. Find the co-ordinates of the point that divides \([PQ]\) internally in the ratio \(3 : 1\).

3 (a)

Point is \(((1(-1)+3(3)) \div 3+1, (1(4)+3(7)) \div 3+1) = (8 \div 4, 25 \div 4) = (2, 6\frac{1}{4})\)

*Note: General Guideline 8 does not necessarily apply here

**Blunders (-3)**

B1 Incorrect ratio formula  
B2 Incorrect translation

**Slips (-1)**

S1 Arithmetic errors

**Attempts (5 marks)**

A1 One correct ordinate

**Worthless (0 marks)**

W1 Midpoint used once

<table>
<thead>
<tr>
<th>Part (b)</th>
<th>35 (20, 5, 5, 5) marks</th>
<th>Att (7, 2, 2, 2)</th>
</tr>
</thead>
</table>

3 (b) \( f \) is the transformation \((x, y) \rightarrow (x', y')\), where \(x' = x - y\) and \(y' = 2x + 3y\).

- (i) Find the equation of \(f(l_1')\), the image of \(l_1\) under \(f\).

- (ii) Prove that \(f\) maps every pair of parallel lines to a pair of parallel lines.
  You may assume that \(f\) maps every line to a line.

- (iii) The line \(l_2\) is parallel to the line \(l_1\).
  \(f(l_2')\) intersects the \(x\)-axis at \(A'\) and the \(y\)-axis at \(B'\).
  The area of the triangle \(A'O'B'\) is 9 square units, where \(O\) is the origin.
  Find the two possible equations of \(l_2\).

- (iv) Given that \(A' = f(A)\) and \(B' = f(B)\), show that \(|\angle AOB| \neq |\angle A'O'B'|\).

[44]
Part (b)(i)  20 marks  

\[ \begin{align*} 
2x' &= 2x - 2y \\
y' &= 2x + 3y \\
2x' - y' &= -5y \\ 
\Rightarrow y &= \frac{1}{5}(-2x' + y') \\
x &= x' + y \\ 
\Rightarrow x &= x' + \frac{1}{5}(-2x' + y') \\ 
\Rightarrow x &= \frac{1}{5}(3x' + y'). \\
f(l_1) &= \frac{2}{5}(3x' + y') - \frac{1}{5}(-2x' + y') - 1 = 0 \\ 
\Rightarrow 8x' + y' - 5 &= 0.
\end{align*} \]

**Blunders (-3)**
- B1 \( f(l_1) \) not in form \( px' + qy' + r = 0 \) or \( y' = mx' + c \)
- B2 Incorrect matrix
- B3 Incorrect matrix multiplication

**Slips (-1)**
- S1 Arithmetic errors

**Attempts (7 marks)**
- A1 Effort at \( x \) or \( y \) expressed in terms of \( x' \) and \( y' \)
- A2 Correct matrix for \( f \) when finding \( f(l_1) \)
- A3 Correct image point on \( f(l_1) \)

Part (b)(ii)  5 marks  

\[ \begin{align*} 
s_1 : ax + by + c &= 0 \quad \text{and} \quad s_2 : ax + by + d &= 0 \quad \text{are two parallel lines.} \\
f(s_1) &= \frac{a}{5}(3x' + y') + \frac{b}{5}(-2x' + y') + c = 0 \\
&\Rightarrow (3a - 2b)x' + (a + b)y' + 5c = 0. \\
f(s_2) &= \frac{a}{5}(3x' + y') + \frac{b}{5}(-2x' + y') + d = 0 \\
&\Rightarrow (3a - 2b)x' + (a + b)y' + 5d = 0 \\
\text{Coefficients of } x' \text{ and } y' \text{ match, so these are parallel lines.} \\
\end{align*} \]

**OR**

Suppose \( f(s_1) \) and \( f(s_2) \) are not parallel. Then, they have a point in common, say \( P' \).

\( f \) is invertible, so let \( P = f^{-1}(P') \).

\( P' \in f(s_1) \Rightarrow P \in s_1 \quad \text{and} \quad P' \in f(s_2) \Rightarrow P \in s_2. \)

This contradicts \( s_1 \parallel s_2 \), (unless they are identical, in which case so are their images).

**Blunders (-3)**
- B1 \( f(s_1) \) or \( f(s_2) \) not in form \( px' + qy' + r = 0 \) or \( y' = mx' + c \)
- B2 Incorrect matrix
- B3 Incorrect matrix multiplication
- B4 Fails to finish correctly

**Slips (-1)**
- S1 Arithmetic errors
Attempts (2 marks)
A1 One image point correct
A2 Specific case e.g. using $2x\!-\!y\!-\!1\!\equiv\!0$ and $2x\!-\!y\!+\!k\!\equiv\!0$
A3 Effort at image of one line only

Part (b) (iii) 5 marks Att 2

3 (b) (iii)

$$f(l_2): 8x' + y' = k. \quad \therefore A' \text{ is } \left(\frac{k}{8}, 0\right) \text{ and } B' \text{ is } (0, k).$$

Area of triangle $A'OB' = \frac{1}{2} \left| \frac{k}{8} \right| = 9.$

\[ \therefore k^2 = 144 \Rightarrow k = \pm 12. \quad \therefore f(l_2): 8x' + y' = \pm 12 \Rightarrow 2x - \frac{y}{3} = 0 \]

Blunders (-3)
B1 One value of $k$
B2 Error in area formula
B3 Fails to find $l_2$ from $f(l_2)$

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 $A'$ or $B'$

Part (b) (iv) 5 marks Att 2

3 (b) (iv)

$$x = \frac{1}{5}(3x' + y') \text{ and } y = \frac{1}{5}(-2x' + y') \text{ and } A' \left(\frac{k}{8}, 0\right), \quad B'(0, k).$$

\[ \therefore A \text{ is } \left(\frac{3k}{40}, \frac{-2k}{40}\right) \text{ and } B \text{ is } \left(\frac{k}{5}, \frac{k}{5}\right). \]

$$\angle A'OB' = 90^\circ.$$

Slope $OA = -\frac{2k}{40} = -\frac{2}{3}$ and slope $OB = \frac{k}{5} = 1 \Rightarrow OA \text{ is not } \perp \text{ to } OB.$

\[ \therefore \angle AOB \neq \angle A'OB' \]

Blunders (-3)
B1 $A$ or $B$ incorrect
B2 Error in slope formula
B3 No conclusion or incorrect conclusion

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Effort to find $A$ or $B$ and stops
A2 Effort at finding angle other than required angle
A3 $\angle A'OB' = 90^0$
QUESTION 4

Part (a) 5 marks Att 2

4 (a) Evaluate \( \lim_{x \to 0} \left( \frac{\sin 2x + \sin x}{3x} \right) \).

\[
\begin{align*}
\lim_{x \to 0} \left( \frac{\sin 2x + \sin x}{3x} \right) &= \lim_{x \to 0} \left( \frac{2 \sin x \cos x + \sin x}{3x} \right) = \lim_{x \to 0} \left( \frac{\sin x (2 \cos x + 1)}{3x} \right) = \frac{1}{3} \cdot 1 \cdot 2 = \frac{2}{3}.
\end{align*}
\]

OR

\[
\begin{align*}
\lim_{x \to 0} \left( \frac{\sin 2x + \sin x}{3x} \right) &= \lim_{x \to 0} \left( \frac{\sin 2x}{3x} \right) + \lim_{x \to 0} \left( \frac{\sin x}{x} \right) = \frac{2}{3} + 1 = \frac{2}{3} + 1 = \frac{5}{3}.
\end{align*}
\]

Blunders (-3)
B1 Error rewriting as sum of two limits
B2 Error in \( \sin 2x \) as a product of two functions
B3 Mishandling \( \frac{\sin \theta}{\theta} \)

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Correct answer without work
A2 Correct factors
4 (b) Find all the solutions of the equation
\[ \sin 2x + \cos x = 0, \text{ where } 0^\circ \leq x \leq 360^\circ. \]

Transform equation 10 marks
Solve for \( \cos/sin \) 10 marks
Solutions 10 marks

\[
\begin{align*}
\sin 2x + \cos x &= 0 \\
2\sin x \cos x + \cos x &= 0 \quad \Rightarrow \quad \cos(2\sin x + 1) = 0. \\
\therefore \quad \cos x &= 0 \quad \text{or} \quad \sin x = -\frac{1}{2}. \\
\therefore \quad x &= 90^\circ, 270^\circ \quad \text{or} \quad x = 210^\circ, 330^\circ. \\
\text{Solution} &= \{90^\circ, 210^\circ, 270^\circ, 330^\circ\}.
\end{align*}
\]

Blunders (-3)
B1 Error in expansion of \( \sin 2x \)
B2 Error in factors
B3 Error in roots
B4 Missing and/or incorrect solutions (to a max of 3)
B5 Solutions outside the range (to a max of 3)

Slips (-1)
S1 Arithmetic errors

Attempts(3, 3, 3)
A1 \( \sin x \cos x + \cos x = 0 \) and stops
A2 One correct angle

Part (c) 15 (5, 5, 5) marks

4 (c) The diagram shows two concentric circles. A tangent to the inner circle cuts the outer circle at \( B \) and \( C \), where \( |BC| = 2x \).

(i) Express the area of the shaded region in terms of \( x \).

(ii) In the case where the radius of the outer circle is \( 2x \), show that the portion of the shaded region that lies below \( BC \) has area \( \left(\frac{2\pi}{3} - \sqrt{3}\right)x^2 \).
Part (c) (i) Area in terms of radii  
5 marks  
Area in terms of \( x \)  
5 marks

4 (c) (i) 
Let radius of large circle = \( R \) and radius of small circle = \( r \).
Shaded region = \( \pi R^2 - \pi r^2 = \pi (R^2 - r^2) \)
But \( R^2 = x^2 + r^2 \Rightarrow R^2 - r^2 = x^2 \).
\( \therefore \) Shaded region = \( \pi x^2 \).

Blunders (-3)
B1 Area = \( \pi r^2 - \pi R^2 \) or \( \pi R^2 + \pi r^2 \)
B2 Incorrect value of \( x \) for bisected chord
B3 Incorrect use of Pythagoras

Slips (-1)
S1 Arithmetic errors

Attempts (2, 2 marks)
A1 Bisector of chord indicated

Part (c) (ii)  
5 marks

4 (c) (ii) 
\[ \sin \angle BOD = \frac{x}{2x} = \frac{1}{2} \Rightarrow |\angle BOD| = \frac{\pi}{6} \Rightarrow |\angle BOC| = \frac{\pi}{3}. \]
\( \therefore \) Required area = area of sector \( BOC \) - area of triangle \( BOC \).
\[ = \frac{1}{2} r^2 \theta - \frac{1}{2} |BC||OD| \]
\[ = \frac{1}{2} (2x)^2 \left( \frac{\pi}{3} \right) - \frac{1}{2} (2x)(\sqrt{3}x), \quad [|OD| = \sqrt{3}x] \]
\[ = \frac{2x^2 \pi}{3} - x^2 \sqrt{3} = \left( \frac{2\pi}{3} - \sqrt{3} \right)x^2. \]

Blunders (-3)
B1 \( \angle BOC \) incorrect
B2 Incorrect radius substituted into sector formula
B3 Incorrect use of Pythagoras i.e \( |OD| \) incorrect
B4 Incorrect conclusion stated or implied

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Area of sector with some substitution
A2 Required area identified
QUESTION 5

Part (a) 10 marks Att 3
Part (b) 15 (5, 5, 5) marks Att (2, 2, 2)
Part (c) 25 (10, 10, 5) marks Att (3, 3, 2)

5 (a) Find the values of \( x \) for which \( 3 \tan x = \sqrt{3} \), where \( 0^\circ \leq x \leq 360^\circ \).

\[
3 \tan x = \sqrt{3} \quad \Rightarrow \quad \tan x = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.
\]

\[\therefore \ x = 30^\circ, 210^\circ.\]

**Blunders (-3)**

B1. Mishandling \( \frac{\sqrt{3}}{3} \)
B2. Each incorrect angle and/or omitted angle
B3. Each incorrect angle outside the range

**Slips (-1)**

S1. Arithmetic errors

**Attempts (3 marks)**

A1. One correct angle without work

Part (b) 15 (5, 5, 5) marks Att (2, 2, 2)

5 (b) (i) Prove that \( \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \).

\[ \begin{align*}
\tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\
&= \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B} \\
&= \frac{\tan A + \tan B}{1 - \tan A \tan B}.
\end{align*} \]

**Blunders (-3)**

B1. Error in expanding \( \sin(A+B) \)
B2. Error in expanding \( \cos(A+B) \)
B3. \( \sin A \cos B + \cos A \sin B = \sin(A+B) \) or equivalent not stated
Show that if $\alpha + \beta = 90^\circ$, then $\frac{\tan 2\alpha}{\tan 2\beta} = -1$.

\[
\frac{\tan 2\alpha}{\tan 2\beta} = \frac{\tan 2\alpha}{\tan(180^\circ - 2\alpha)} = \frac{\tan 2\alpha}{-\tan 2\alpha} = -1.
\]

Blunders (-3)
B1 Error in $\tan(180^0 - 2\alpha)$ expansion
B2 Incorrect conclusion

Attempts (2 marks)
A1 $\beta = 90^0 - \alpha$ or $2\beta = 180^0 - 2\alpha$ and stops

A and B are two helicopter landing pads on level ground. C is another point on the same level ground. $|BC| = 800$ metres, $|AC| = 900$ metres, and $\angle BCA = 60^\circ$.

A helicopter is hovering vertically above A. A person at C observes the helicopter to have an angle of elevation of $30^\circ$.

(i) Find $|AD|$, in surd form.

(ii) Find $|BD|$.
### Part (c) (i) 10 marks Att 3

\[ \tan 30^\circ = \frac{AD}{900} \Rightarrow |AD| = 900 \left( \frac{1}{\sqrt{3}} \right) = 300\sqrt{3} \text{ m.} \]

**Blunders (-3)**
- B1 Incorrect use of trigonometric ratio
- B2 Answer not in surd form

**Slips (-1)**
- S1 Arithmetic errors
- S2 Units omitted or incorrect

**Attempts (3 marks)**
- A1 Identifies relevant right angled triangle

**Worthless (0 marks)**
- W1 Relevant right angled triangle not indicated or implied

### Part (c) (ii) \[ |AB|^2 \] 10 marks Att 3

\[ |BD|^2 \] 5 marks Att 2

\[
\begin{align*}
|AB|^2 &= (800)^2 + (900)^2 - 2(800)(900)\cos 60^\circ \\
&= 640000 + 810000 - 720000 = 730000 \\

|BD|^2 &= |AB|^2 + |AD|^2 = 730000 + 270000 = 1000000. \\
\end{align*}
\]

\[ \therefore |BD| = 1000 \text{ m.} \]

*Accept candidate’s answer from (c)(i)*
*If \( |AB|^2 \) worthless, then attempt at most for remainder of section*

**Blunders (-3)**
- B1 Error in cosine formula with substitution
- B2 Use of decimals leading to incorrect answer

**Slips (-1)**
- S1 Arithmetic errors
- S2 Units omitted or incorrect (apply once only in this section)

**Attempts (3, 2 marks)**
- A1 Cosine formula with some correct substitution

**Worthless (0 marks)**
- W1 Right angle not identified or indicated
### QUESTION 6

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (10, 10) marks</td>
<td>Att (3, 3)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

#### Part (a) 10 marks Att 3

6 (a) Two adults and four children stand in a row for a photograph. How many different arrangements are possible if the four children are between the two adults?

Blunders (-3)
B1 Error in setting up quadratic
B2 Error in solving quadratic
B3 Error in general term
B4 Equation in $l$ and $k$

Attempts (3 Attempts)
A1 4!
A2 21+4! or 2+4! (with or without further work)

Worthless (0 marks)
W1 6!

#### Part (b) 20 (10, 10) marks Att (3, 3)

6 (b) (i) Solve the difference equation $u_{n+2} - 6u_{n+1} + 8u_n = 0$, where $n \geq 0$, given that $u_0 = 0$ and $u_1 = 4$.

(ii) For what value of $n$ is $u_n = 30(2^n)$?

Blunders (-3)
B1 Error in setting up quadratic
B2 Error in solving quadratic
B3 Error in general term
B4 Equation in $l$ and $k$

Slips (-1)
S1 Arithmetic errors
Attempts (3 marks)
A1 Substitution into quadratic formula
A2 Equation in \( l \) and \( k \)

---

**Part (b) (ii)**

10 marks

\[
6 \text{ (b) (ii)}
\]

\[
2^{2n+1} - 2^{n+1} = 30(2^n) \Rightarrow 2^n \cdot 2^1 \cdot 2^n - 2^n = 30 \cdot 2^2 \Rightarrow 2^n \cdot 2 - 2 = 30
\]

\[
\Rightarrow 2^n - 1 = 15 \Rightarrow 2^n = 16 \Rightarrow n = 4.
\]

Blunders (-3)
B1 Error in handling indices

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 \( 2^{2n+1} = 2^{2n} \cdot 2 \) or equivalent

---

**Part (c)**

20 (5, 5, 5, 5) marks

**Part (c) (i)**

5 marks

\[
6 \text{ (c) (i)}
\]

\[
P(\text{five diamonds}) = \frac{13C_5}{52C_5} = \frac{1287}{259860} = 4.95 \times 10^{-4} = 0.000495 \text{ or } 0.000502598960
\]

Blunders (-3)
B1 Incorrect number of favourable outcomes
B2 Incorrect number of possible outcomes

Slips (-1)
S1 Answer not to two significant figures

Attempts (2 marks)
A1 \( \frac{13C_5}{52C_5} \)

---

[54]
Part (c) (ii) 5 marks

<table>
<thead>
<tr>
<th>6 (c) (ii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ P(\text{all same suit}) = P(5 \text{ diamonds}) + P(5 \text{ hearts}) + P(5 \text{ clubs}) + P(5 \text{ spades}) ]</td>
</tr>
<tr>
<td>[ = 4 \times \frac{{^{13}C_5}}{^{52}C_5} = \frac{5148}{2598960} = 1.98 \times 10^{-3} = 2.0 \times 10^{-3} \text{ or } 0.0020 ]</td>
</tr>
</tbody>
</table>

**Blunders (-3)**
- B1 Incorrect number of favourable outcomes
- B2 Incorrect number of possible outcomes

**Slips (-1)**
- S1 Answer not to two significant figures

**Attempts (2 marks)**
- A1 \[ 4 \times \frac{{^{13}C_5}}{^{52}C_5} \]

Part (c) (iii) 5 marks

<table>
<thead>
<tr>
<th>6 (c) (iii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ P(\text{ace, 2, 3, 4, 5 of diamonds}) = \frac{{^{5}C_5}}{^{52}C_5} ]</td>
</tr>
<tr>
<td>[ = \frac{1}{2598960} = 3.84 \times 10^{-7} = 3.8 \times 10^{-7} ]</td>
</tr>
<tr>
<td>or [ 0.0000038 ]</td>
</tr>
</tbody>
</table>

**Blunders (-3)**
- B1 Incorrect number of favourable outcomes
- B2 Incorrect number of possible outcomes

**Slips (-1)**
- S1 Answer not to two significant figures

**Attempts (2 marks)**
- A1 \[ \frac{{^{5}C_5}}{^{52}C_5} \]

Part (c) (iv) 5 marks

<table>
<thead>
<tr>
<th>6 (c) (iv)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ P(\text{four aces}) = \frac{{^{4}C_4 \times ^{48}C_1}}{^{52}C_5} = \frac{48}{2598960} = 1.84 \times 10^{-5} = 1.8 \times 10^{-5} \text{ or } 0.000018 ]</td>
</tr>
</tbody>
</table>

**Blunders (-3)**
- B1 Incorrect number of favourable outcomes
- B2 Incorrect number of possible outcomes

**Slips (-1)**
- S1 Answer not to two significant figures

**Attempts (2 marks)**
- A1 \[ \frac{{^{4}C_4 \times ^{48}C_1}}{^{52}C_5} \]
### QUESTION 7

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (10, 10) marks</td>
<td>Att (3, 3)</td>
</tr>
</tbody>
</table>

#### Part (a) 10 (5, 5) marks Att (2, 2)

**7 (a)** A team of four is selected from a group of seven girls and five boys.

(i) How many different selections are possible?

(ii) How many of these selections include at least one girl?

#### Part (a) (i) 5 marks Att 2

**7 (a) (i)**

Number of selections $= \binom{12}{4} = 495$.

**Blunders (-3)**

B1 $7C_4 + 5C_4$

**Slips (-1)**

S1 Arithmetic errors

**Attempts (2 marks)**

A1 $7C_4$ or $5C_4$

**Worthless**

W1 $\frac{12!}{4!}$

#### Part (a) (ii) 5 marks Att 2

**7 (a) (ii)**

Number of selections with no girl $= 5C_4 = 5$.

Number of selections with at least one girl $= 495 - 5 = 490$.

**OR**

$7C_1 \cdot 7C_3 + 7C_2 \cdot 5C_2 + 7C_3 \cdot 5C_1 + 7C_4 \cdot 5C_0 = 490$

**Blunders (-3)**

B1 Term omitted

B2 Incomplete answer

**Slips (-1)**

S1 Arithmetic errors

**Attempts (2 marks)**

A1 $7C_4$

A2 $7C_1 \cdot 5C_3$ or equivalent

[56]
Part (b)  20 (5, 5, 5, 5) marks  Att (2, 2, 2, 2)

7 (b) A marble falls down from A and must follow one of the path indicated on the diagram. All paths from A to the bottom row are equally likely to be followed.

(i) One of the paths from A to H is A-B-D-H. List the other two possible paths from A to H.

(ii) Find the probability that the marble passes through H or J.

(iii) Find the probability that the marble lands on N.

(iv) Two marbles fall from A, one after the other, without affecting each other. Find the probability that they both land at P.

---

**Part (b) (i)**  5 marks  Att 2

7 (b) (i)

There are two other possible paths: A-B-E-H and A-C-E-H.

*Blunders (-3)*

B1 One path only

**Part (b) (ii)**  5 marks  Att 2

7 (b) (ii)

Paths to J are A-B-E-J, A-C-E-J and A-C-F-J.

∴ There are 6 paths from A to H or J.

All of the possible paths from A to the GHJK row are:


∴ There are 8 possible paths.

(Or just $2 \times 2 \times 2 = 8$.)

∴ Probability $\frac{6}{8} = \frac{3}{4}$.  

*Blunders (-3)*

B1 Number of favourable outcomes incorrect

B2 Number of possible outcomes incorrect

*Slips (-1)*

S1 Arithmetic errors

**Attempts (2 marks)**

A1 Favourable and/or all possible outcomes listed correctly

**Worthless (0 marks)**

W1 Incomplete list of outcomes and stops
Part (b) (iii) 5 marks Att 2

7 (b) (iii)

6 paths to N: ABDHN, ABEHN, ABEJN, ACEHN, ACEJN, ACFJN.
16 possible paths from A to bottom row.
∴ Probability = \frac{6}{16} = \frac{3}{8}.

Blunders (-3)
B1 Number of favourable outcomes incorrect
B2 Number of possible outcomes incorrect

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Favourable and/or all possible outcomes listed correctly

Worthless (0 marks)
W1 Incomplete list of outcomes and stops
W2 \frac{1}{5} with or without explanation

Part (b) (iv) 5 marks Att 2

7 (b) (iv)

There are four paths from A to P. ∴ 4×4 outcomes of interest
There are 16 possible paths for each marble. ∴ 16×16 outcomes in total.
∴ Probability = \frac{4×4}{16×16} = \frac{1}{16}.

Blunders (-3)
B1 Number of favourable outcomes incorrect
B2 Number of possible outcomes incorrect

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Favourable and/or all possible outcomes listed correctly
A2 One marble only

Worthless (0 marks)
W1 \frac{1}{5} × \frac{1}{5} = \frac{1}{25}

Part (c) 20 (10, 10) marks Att (3, 3)

7 (c) The real numbers \(a\), \(b\) and \(c\) have mean \(\mu\) and standard deviation \(\sigma\).

(i) Show that the mean of the numbers \(\frac{a-\mu}{\sigma}\), \(\frac{b-\mu}{\sigma}\) and \(\frac{c-\mu}{\sigma}\) is 0.

(ii) Find, with justification, the standard deviation of the numbers \(\frac{a-\mu}{\sigma}\), \(\frac{b-\mu}{\sigma}\) and \(\frac{c-\mu}{\sigma}\).
Part (c) (i)  

\[ \text{Mean} = \frac{a - \mu + b - \mu + c - \mu}{\sigma} = \frac{a + b + c - 3\mu}{3\sigma} = \frac{3\mu - 3\mu}{3\sigma} = 0, \text{ as } \frac{a + b + c}{3} = \mu. \]

**Blunders (-3)**

B1 \( a + b + c \neq 3\mu \) or equivalent

**Slips (-1)**

S1 Arithmetic errors

**Attempts (3 marks)**

A1 Correct mean of \( a, b, \) and \( c \)

A2 Expression for mean of \( \frac{a - \mu}{\sigma}, \frac{b - \mu}{\sigma} \) and \( \frac{c - \mu}{\sigma} \)

**Worthless (0 Marks)**

W1 \( \frac{a - \mu}{\sigma} + \frac{b - \mu}{\sigma} + \frac{c - \mu}{\sigma} \) and stops

Part (c) (ii)  

\[ \text{The numbers } a, b \text{ and } c \text{ have mean } \mu \text{ and standard deviation } \sigma. \]

\[ \therefore \sigma = \sqrt{\frac{(a - \mu)^2 + (b - \mu)^2 + (c - \mu)^2}{3}}. \]

The numbers \( \frac{a - \mu}{\sigma}, \frac{b - \mu}{\sigma} \) and \( \frac{c - \mu}{\sigma} \), with mean = 0, has standard deviation

\[ \sqrt{\frac{\left(\frac{a - \mu}{\sigma} - 0\right)^2 + \left(\frac{b - \mu}{\sigma} - 0\right)^2 + \left(\frac{c - \mu}{\sigma} - 0\right)^2}{3}} = \frac{1}{\sigma} \sqrt{\frac{(a - \mu)^2 + (b - \mu)^2 + (c - \mu)^2}{3}} = \frac{1}{\sigma}(\sigma) = 1 \]

**Blunders (-3)**

B1 Error in squaring

**Slips (-1)**

S1 Arithmetic errors

**Attempts (3 marks)**

A1 Expression for standard deviation correct
QUESTION 8

Part (a) 15 marks Att 5

8 (a) Use integration by parts to find \( \int x \sin x \, dx \).

Part (a) 15 marks Att 5

\[
\int udv = uv - \int vdu.
\]

Let \( u = x \Rightarrow du = dx \) and \( dv = \int \sin x \, dx \Rightarrow v = -\cos x \).

\[
\therefore \int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + \text{constant of integration}.
\]

Blunders (-3)
B1 Incorrect differentiation or integration
B2 Incorrect ‘parts’ formula

Slips (-1)
S1 Arithmetic error
S2 Omits constant of integration

Attempts (5 marks)
A1 One correct assigning in ‘parts’ formula
A2 Correct relevant differentiation or integration

Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

8 (b) A window is in the shape of a rectangle with a semicircle on top.
The radius of the semicircle is \( r \) metres and the height of the
rectangular part is \( x \) metres.
The perimeter of the window is 20 metres.

(i) Use the perimeter to express \( x \) in terms of \( r \) and \( \pi \).

(ii) Find, in terms of \( \pi \), the value of \( r \) for which the area of
the window is a maximum.

Part (b) (i) 5 marks Att 2

8 (b) (i) Perimeter = \( 2x + 2r + \pi r = 20 \) \( \Rightarrow x = \frac{20 - 2r - \pi r}{2} \) metres.
**Blunders (-3)**
B1  Error in perimeter
B2  Answer not in required form

**Slips (-1)**
S1  Arithmetic errors
S2  Omits units or incorrect units

**Attempts (2 marks)**
A1  Expression for perimeter of semicircle
A2  Expression for perimeter of rectangular section of window

<table>
<thead>
<tr>
<th>Part (b) (ii)</th>
<th>Area in terms of ( r )</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differentiation</td>
<td>5 marks</td>
<td>Att 2</td>
<td></td>
</tr>
<tr>
<td>Finish</td>
<td>5 marks</td>
<td>Att 2</td>
<td></td>
</tr>
</tbody>
</table>

**8 (b) (ii)**

Area of window \( A = 2rx + \frac{1}{2}\pi r^2 \).

\[
\therefore A = 2r \left( \frac{20 - 2r - \pi r}{2} \right) + \frac{1}{2}\pi r^2 = 20r - 2r^2 - \frac{1}{2}\pi r^2.
\]

\[
\therefore \frac{dA}{dr} = 20 - 4r - \pi r.
\]

For \( \frac{dA}{dr} = 0 \Rightarrow 20 - 4r - \pi r = 0 \)

\[\Rightarrow r(4 + \pi) = 20.\]

\[\therefore r = \frac{20}{4 + \pi}.\]

\[
\frac{d^2A}{dr^2} = -4 - \pi < 0. \quad \therefore \text{Area of window is a maximum for } r = \frac{20}{4 + \pi} \text{ metres}
\]

* If candidate’s expression for perimeter in (b)(i) contains square units, then cannot get any further marks in this section

**Blunders (-3)**
B1  Error in eliminating \( x \) from expression for area
B2  Error in differentiation
B3  Error in finding \( r \)

**Slips (-1)**
S1  Arithmetic errors
S2  Omits units or incorrect units

**Attempts (2, 2, 2)**
A1  Some correct differentiation
A2  \( 20 - 4r - \pi r = 0 \) and stops

**Worthless (0 marks)**
W1  Non quadratic expression for area
The Maclaurin series for \( \tan^{-1} x \) is \( x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \)

(i) Write down the general term of the series.

(ii) Use the Ratio Test to show that the series converges for \( |x| < 1 \).

(iii) Using the fact that \( \frac{\pi}{4} = 4\tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} \), and taking the first three terms in the Maclaurin series for \( \tan^{-1} x \), find an approximation for \( \pi \).

Give your answer correct to five decimal places.

\[
8 (c) (i) \quad u_n = \frac{x^{2n-1}}{2n-1} \left( -1 \right)^{n+1}
\]

Blunders (-3)
B1 \(-1\) omitted in general term
B2 Incorrect \( x \) index in general term
B3 Value of \( n \) in denominator does not match index of \( x \) in numerator

Slips (-1)

Attempts (2 marks)
A1 One part of general term correct

\[
\begin{align*}
\text{(c) (ii)} & \quad \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \frac{x^{2n+1}}{2n+1} \left( -1 \right)^{n+2} \times \frac{2n-1}{x^{2n-1} \left( -1 \right)^{n+1}} \\
& = \lim_{n \to \infty} \left| \frac{x^2 (2n-1)}{2n+1} \right| = \lim_{n \to \infty} \left| \frac{x^2 \left( 2 - \frac{1}{n} \right)}{2 + \frac{1}{n}} \right| = x^2.
\end{align*}
\]

\[\therefore \text{ Convergent for } x^2 < 1 \Rightarrow \text{ convergent for } |x| < 1.\]

*Note: If candidate gets 0 marks in (c)(i) then attempt mark at most in (c)(ii)

If candidate’s \( x \) index is incorrect in (c)(i), then attempt mark at most in (c)(ii)

Blunders (-3)
B1 Error in \( u_{n+1} \)
B2 Error in limits other than slips
B3 \( |x^2| \) or \( -|x^2| \) mishandled
B4 Incorrect conclusion

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Ratio test used correctly
\[
\frac{\pi}{4} = 4 \left[ \frac{1}{5} - \frac{1}{3(5)^3} + \frac{1}{5(5)^3} \right] - \left[ \frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^3} \right]
\]

\[\therefore \pi = 3.14162.\]

**Blunders (-3)**

B1 Term omitted in expansion

**Slips (-1)**

S1 Arithmetic error

**Attempts (2 marks)**

A1 Correct listing of one series and stops
QUESTION 9

Part (a) 10 marks Att 3
Part (b) 20 (10, 10) marks Att (3, 3)
Part (c) 20 (5, 5, 10) marks Att (2, 2, 3)

Part (a) 10 marks Att 3

9 (a) Z is a random variable with standard normal distribution.
Use the tables to find the value of $z_1$ for which $P(Z \geq z_1) = 0.0778$.

\[
P(Z \geq z_1) = 0.0778 \implies 1 - P(Z \leq z_1) = 0.0778.
\]

\[
P(Z \leq z_1) = 0.9222 \implies z_1 = 1.42.
\]

Blunders (-3)
B1 Incorrect reading of tables
B2 Incorrect area

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 $P(Z \geq z_1) \implies 1 - P(Z \leq z_1)$

Part (b) 20 (10, 10) marks Att (3, 3)

9 (b) A die is biased in such a way that the probability of rolling a six is $p$.
The other five numbers are equally likely. This biased die and a fair die are rolled simultaneously. Show that the probability of rolling a total of 7 is independent of $p$.

Probability of 6 on biased die = $p$
Probability of not 6 on biased die = $1-p$

$\implies$ probability of any other single outcome (of which there are 5) on die $= \frac{1-p}{5}$.

Probability of a total of seven from biased and fair die
\[
\begin{align*}
\{i.e. (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)\}
\implies &\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6} \\
= &\frac{p}{6} + \frac{5(1-p)}{6} = \frac{p + 1-p}{6} = \frac{1}{6}.
\end{align*}
\]

Blunders (-3)
B1 Divisor other than 5
B2 Each term omitted to max of 3
B3 Incorrect or no conclusion written or implied
The mean percentage mark for candidates in the 2010 Leaving Certificate Higher Level Mathematics examination was 67.0%, with a standard deviation of 10.4%. The suggestion that candidates who appealed their results have, on average, similar results to all other candidates, is being investigated. A random sample of candidates who appealed is taken. The mean percentage mark of this sample is 69.3%.

(i) Show that if the sample size was 25, then this result is not significant at the 5% level.

(ii) Show that if the sample size was 100, then this result is significant at the 5% level.

(iii) What is the smallest sample size for which this result could be regarded as significant at the 5% level?

\[
\frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{69.3 - 67}{10.4 / \sqrt{25}} = 2.3 < 1.96.
\]

\[
\therefore \text{Result not significant.}
\]

OR

\[
\mu - 1.96\sigma / \sqrt{n} = \bar{x} 
\mu + 1.96\sigma / \sqrt{n}
\]

\[
67 - \frac{1.96(10.4)}{\sqrt{25}} \leq \bar{x} \leq 67 + \frac{1.96(10.4)}{\sqrt{25}}
\]

\[
62.9232 \leq \bar{x} \leq 71.0768
\]

Within range \(\Rightarrow\) not significant

Blunders (-3)

B1 Error in formula

B2 \(\sigma / \sqrt{n} \neq \sigma / \sqrt{n}\)

B3 Incorrect or no conclusion implied

Slips (-1)

S1 Arithmetic errors

Attempts (3, 3 marks)

A1 Reference to \(1 - p\)

A2 Listing favourable outcomes (must have (6, 1) and at least one other outcome)

A3 One correct term

Part (c) 20 (5, 5, 10) marks Att (2, 2, 3)

9 (c) The mean percentage mark for candidates in the 2010 Leaving Certificate Higher Level Mathematics examination was 67.0%, with a standard deviation of 10.4%. The suggestion that candidates who appealed their results have, on average, similar results to all other candidates, is being investigated. A random sample of candidates who appealed is taken. The mean percentage mark of this sample is 69.3%.

(i) Show that if the sample size was 25, then this result is not significant at the 5% level.

(ii) Show that if the sample size was 100, then this result is significant at the 5% level.

(iii) What is the smallest sample size for which this result could be regarded as significant at the 5% level?
Part (c) (ii) 5 marks Att 2

9 (c) (ii)

\[ n = 100, \ \mu = 67, \ \sigma = 10.4, \ \bar{x} = 69.3. \]

\[ \frac{69.3 - 67}{\sqrt{10.4}} = \frac{2.3}{1.04} = 2.11 > 1.96. \]

\[ \therefore \text{Result is significant.} \]

Blunders (-3)
B1 Error in formula
B2 \( \sigma_s \neq \frac{\sigma}{\sqrt{n}} \)
B3 Incorrect or no conclusion

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Formula partially substituted

Part (c) (iii) 10 marks Att 3

9 (c) (iii)

\[ \mu = 67, \ \sigma = 10.4, \ \bar{x} = 69.3. \]

\[ \frac{69.3 - 67}{\sqrt{10.4}} = \frac{2.3\sqrt{n}}{10.4} \geq 1.96. \]

\[ 2.3\sqrt{n} \geq 196 \times 10.4 \Rightarrow \sqrt{n} \geq 8.862. \]

\[ \therefore n > 78.55 \Rightarrow n = 79. \]

\[ \therefore \text{Smallest sample size is 79.} \]

Blunders (-3)
B1 Error in formula
B2 \( \sigma_s \neq \frac{\sigma}{\sqrt{n}} \)
B3 Incorrect or smallest sample not chosen

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 Formula partially substituted
QUESTION 10

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 40 (5, 5, 5, 5, 10, 5, 5) marks Att (2, 2, 2, 2, 3, 2, 2)

Part (a) 10 (5, 5) marks Att (2, 2)

10 (a) A Cayley table for the group \( \{a, b, c\}, * \) is shown. 

\[
\begin{array}{c|cccc}
* & a & b & c \\
\hline
a & c & a & b \\
b & a & b & c \\
c & b & c & a
\end{array}
\]

(i) Write down the identity element. 

(ii) Write down the inverse of each element.

Part (a) (i) 5 marks Att 2

10 (a) (i) 

Identity element = b.

Attempts (2 marks)
A1 Identity property stated but element not identified

Part (a) (ii) 5 marks Att 2

10 (a) (ii) 

\[a^{-1} = c, \quad b^{-1} = b, \quad c^{-1} = a.\]

Blunders (-3)
B1 Inverse of any element omitted

Attempt (2 marks)
A1 \(a^* a^{-1} = b\)
A2 Any correct inverse

Part (b) 40 (5, 5, 5, 5, 10, 5, 5) marks Att (2, 2, 2, 2, 3, 2, 2)

10 (b) A regular tetrahedron has twelve rotational symmetries. These form a group under composition, \( \circ \). The symmetries can be represented as permutations of the vertices \( A, B, C \) and \( D \).

(i) Write down in permutation form, one element \( x \) of order 3, and describe this symmetry geometrically.

(ii) Write down in permutation form, one element \( y \) of order 2, and describe this symmetry geometrically

(iii) Show that \( x \circ y \neq y \circ x \).

(iv) Let \( S \) be the set \( \{e, x, y, x \circ y, y \circ x, x \circ x\} \), where \( e \) is the identity transformation. Show that \( S \) is not closed under \( \circ \).

(v) Let \( H \) be a subgroup of \( G \). Let \( x \in H \) and \( y \in H \). Show that \( H = G \).
Part (b) (i) Permutation 5 marks  
Description 5 marks  

10 (b) (i) Fix one vertex e.g. A  
There are eight possible answers, such as:  

\[ x = \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix} \]

Geometrically, this is a rotation of \( \frac{2\pi}{3} \) about the axis \( AG \), where \( G \) is the centroid of the triangle \( BCD \).  
The other solutions correspond to rotations of \( \frac{2\pi}{3} \) or \( \frac{4\pi}{3} \) about this or similar axes.

Blunders (-3)  
B1 Permutation other than order 3  
B2 Incomplete geometrical justification

Slips (-1)  
S1 Arithmetic errors

Attempts (2, 2 marks)  
A1 Incorrect angle of rotation

Part (b) (ii) Permutation 5 marks  
Interpretation 5 marks  

10 (b) (ii)  
There are three possible answers, such as:  

\[ y = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} \]

Geometrically, this is a rotation of \( \pi \) about the axis through the mid points of the opposite edges \([AD]\) and \([BC]\).

Blunders (-3)  
B1 Incomplete geometrical interpretation

Slips (-1)  
S1 Arithmetic errors

Attempt (2, 2 marks)  
A1 Reference to \( \pi \)

Part (b) (iii) 10 marks  

10 (b) (iii)  
\( x \circ y = \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix} \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ B & D & C & A \end{pmatrix} \)  
\( y \circ x = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ D & B & A & C \end{pmatrix} \)  
\[
\therefore x \circ y \neq y \circ x.
\]

Note: compositions depend on candidate’s choice of \( x \) and \( y \), but will be unequal in all correct cases.
Blunders (-3)
B1  Error in composition
B2  Incorrect conclusion stated or implied

Slips (-1)
S1  Arithmetic errors

Attempts (3 marks)
A1  $x \circ y$ identified

<table>
<thead>
<tr>
<th>Part (b) (iv)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 (b) (iv)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(y \circ x)(x \circ y) = \begin{pmatrix} A &amp; B &amp; C &amp; D \ D &amp; B &amp; A &amp; C \end{pmatrix} \begin{pmatrix} A &amp; B &amp; C &amp; D \ B &amp; D &amp; C &amp; A \end{pmatrix} = \begin{pmatrix} A &amp; B &amp; C &amp; D \ B &amp; C &amp; A &amp; D \end{pmatrix} \notin S.$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\therefore S$ is not closed. Note: other correct examples of non-closure exist, and are dependent on candidate’s choice of $x$ and $y.$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Blunders (-3)
B1  Incorrect composition
B2  No conclusion stated or implied

Slips (-1)
S1  Arithmetic errors

Attempts (2 marks)
A1  At least 2 elements of composition correct

<table>
<thead>
<tr>
<th>Part (b) (v)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 (b) (v)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>By Lagrange’s theorem, any subgroup $H$ of $G$ must be of order 1, 2, 3, 4, 6 or 12. But $H$ must at least contain the elements ${ e, x, y, x \circ y, y \circ x, x \circ x }.$ But by part (iii), this set is not closed. Thus it must contain 12 elements. Hence $H = G.$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Blunders (-3)
B1  Error in use of Lagrange’s Theorem
B2  No reference to issue of non-closure from (iii)

Slips (-1)
S1  Arithmetic errors

Attempts (2 marks)
A1  Definition of a subgroup written or implied.
QUESTION 11

Part (a)        10 marks  Att 3
11 (a) An ellipse, centre (0, 0), has eccentricity $\frac{1}{2}$. One focus is at (2,0). Find the equation of the ellipse.

11 (a)

\[ ae = 2 \Rightarrow a\left(\frac{1}{2}\right) = 2 \Rightarrow a = 4 \quad \text{and} \quad b^2 = a^2\left(1-e^2\right) = 16\left(1-\frac{1}{4}\right) = 12. \]

Ellipse is \[ \frac{x^2}{16} + \frac{y^2}{12} = 1. \]

**Blunders (-3)**

B1 Values of $a^2$ and $b^2$ found but equation not formed
B2 Error in formula
B3 Mishandling $e^2$

**Slips (-1)**

S1 Arithmetic errors

**Attempts (3 marks)**

A1 $a = 4$ and stops

Part (b)        40 (10, 5, 10, 15) marks  Att (3, 2, 3, 5)
11 (b)(i) $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points such that $x_1 \leq x_2$.

If the slope of $PQ$ is $\tan \theta$, and the length of $[PQ]$ is $d$, express $(x_2 - x_1)$ and $(y_2 - y_1)$ in terms of $d$ and $\theta$.

(ii) Let $f$ be the transformation \( (x, y) \rightarrow (x', y') \), where
\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}.
\]

Show that
\[
\frac{|f(P)f(Q)|}{|PQ|} = \sqrt{(2\cos\theta + 5\sin\theta)^2 + (3\cos\theta + 4\sin\theta)^2}.
\]

(iii) Deduce that the ratio of lengths on parallel lines is invariant under $f$. 

[70]
Part (b) (i) 10 marks

11 (b) (i) \[ PR = x_2 - x_1 \text{ and } QR = y_2 - y_1. \]
\[
\cos \theta = \frac{x_2 - x_1}{d} \quad \Rightarrow \quad x_2 - x_1 = d \cos \theta.
\]
\[
\sin \theta = \frac{y_2 - y_1}{d} \quad \Rightarrow \quad y_2 - y_1 = d \sin \theta.
\]

Blunders (-3)
B1 Error in trigonometric formula
B2 \( x_2 - x_1 = d \cos \theta \) only

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 \( \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} \)

Part (b) (ii) \[ \frac{f(P)f(Q)}{|PQ|} \] 5 marks

Finish 10 marks

Part (b) (ii)

11 (b) (ii) \[
f(P) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x_1 + \begin{pmatrix} 5 \\ 6 \end{pmatrix} y_1 = \frac{2x_1 + 5y_1 + 6}{3x_1 + 4y_1 + 1} \text{ and } f(Q) = \begin{pmatrix} 2x_2 + 5y_2 + 6 \\ 3x_2 + 4y_2 + 1 \end{pmatrix}.
\]
\[
\therefore \frac{f(P)f(Q)}{|PQ|} = \frac{\sqrt{(2x_2 + 5y_2 + 6 - 2x_1 - 5y_1 - 6)^2 + (3x_2 + 4y_2 + 1 - 3x_1 - 4y_1 - 1)^2}}{d}
\]
\[
= \frac{\sqrt{2(x_2 - x_1) + 5(y_2 - y_1))^2 + [3(x_2 - x_1) + 4(y_2 - y_1)]^2}}{d}
\]
\[
= \frac{\sqrt{(2\cos \theta + 5\sin \theta)^2 + (3\cos \theta + 4\sin \theta)^2}}{d}
\]
\[
= \frac{d\sqrt{(2\cos \theta + 5\sin \theta)^2 + (3\cos \theta + 4\sin \theta)^2}}{d}
\]
\[
= \sqrt{(2\cos \theta + 5\sin \theta)^2 + (3\cos \theta + 4\sin \theta)^2}
\]

Blunders (-3)
B1 Error in matrix multiplication
B2 Incorrect conclusion

Slips (-1)
S1 Arithmetic errors
Attempts (2, 3 marks)
A1 \( f(P) \) or equivalent
A2 Distance formula with some correct substitution for \( |f(P)f(Q)| \)

Part (b) (iii) 15 marks

11 (b) (iii) \[
\begin{align*}
[PQ] \text{and} [RS] \text{are parallel lines} \\
[PQ] \text{ and } [RS] \text{ are mapped to } [f(P)f(Q)] \text{ and } [f(R)f(S)] \text{ respectively.}
\end{align*}
\]

By part (ii), \( |f(P)f(Q)| = k|PQ| \), where \( k = \sqrt{(2\cos \theta + 5\sin \theta)^2 + (3\cos \theta + 4\sin \theta)^2} \).

Since \( k \) depends only on \( \theta \), it is the same \( k \) for both segments.
\[
\therefore \frac{|f(P)f(Q)|}{|f(R)f(S)|} = \frac{k|PQ|}{k|RS|} = \frac{|PQ|}{|RS|}.
\]

Blunders (-3)
B1 Fails to justify \( |f(R)f(S)| = k|RS| \)
B2 No conclusion or incorrect conclusion

Slips (-1)
S1 Arithmetic errors

Attempt (5 marks)
A1 \( |f(P)f(Q)| = k|PQ| \)
MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAELGE

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronadh ar iar-thóirí nach ngnóthaíonn níos mó ná 75% d’iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú sios.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iar-thóirí a ghnóthaíonn 225 marc nó níos lú, e.g. 198 marc × 5% = 9·9 ⇒ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, ríomhtar an bónas de réir na foirmle [300 – bunmharc] × 15%, agus an marc bónais sin a shlánú sios. In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

<table>
<thead>
<tr>
<th>Bunmharc</th>
<th>Marc Bónais</th>
</tr>
</thead>
<tbody>
<tr>
<td>226</td>
<td>11</td>
</tr>
<tr>
<td>227 – 233</td>
<td>10</td>
</tr>
<tr>
<td>234 – 240</td>
<td>9</td>
</tr>
<tr>
<td>241 – 246</td>
<td>8</td>
</tr>
<tr>
<td>247 – 253</td>
<td>7</td>
</tr>
<tr>
<td>254 – 260</td>
<td>6</td>
</tr>
<tr>
<td>261 – 266</td>
<td>5</td>
</tr>
<tr>
<td>267 – 273</td>
<td>4</td>
</tr>
<tr>
<td>274 – 280</td>
<td>3</td>
</tr>
<tr>
<td>281 – 286</td>
<td>2</td>
</tr>
<tr>
<td>287 – 293</td>
<td>1</td>
</tr>
<tr>
<td>294 – 300</td>
<td>0</td>
</tr>
</tbody>
</table>