Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE MATHS

HIGHER LEVEL

MARKING SCHEME
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MARKING SCHEME
LEAVING CERTIFICATE EXAMINATION 2007

MATHEMATICS – HIGHER LEVEL – PAPER 1

GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates’ work as follows:
   • Blunders - mathematical errors/omissions (-3)
   • Slips - numerical errors (-1)
   • Misreadings (provided task is not oversimplified) (-1).

Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   • any correct, relevant step in a part of a question merits at least the attempt mark for that part
   • if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   • a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The same error in the same section of a question is penalised once only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.
QUESTION 1

Part (a) 10 (5, 5) marks Att (2, 2)

1. (a) Simplify \( \frac{x^2 - xy}{x^2 - y^2} \).

Factors: 5 marks Att 2
Cancellation: 5 marks Att 2

\[
\frac{x^2 - xy}{x^2 - y^2} = \frac{x(x - y)}{(x + y)(x - y)} = \frac{x}{x + y}
\]

**Blunders (-3)**
B1 Factors once only
B2 Indices
B3 Incorrect cancellation

Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

1 (b) Let \( f(x) = x^2 + (k + 1)x - k - 2 \), where \( k \) is a constant.

(i) Find the value of \( k \) for which \( f(x) = 0 \) has equal roots.

(ii) Find, in terms of \( k \), the roots of \( f(x) = 0 \).

(iii) Find the range of values of \( k \) for which both roots are positive.

(i) \( b^2 - 4ac = 0 \) applied: 5 marks Att 2
Finish: 5 marks Att 2
(ii) 5 marks Att 2
(iii) 5 marks Att 2

1 (b)
(i) \( f(x) = 0 \Rightarrow x^2 + (k + 1)x + (-k - 2) = 0 \)

Equal roots: \( b^2 - 4ac = 0 \)
\[
(k + 1)^2 - 4(1)(-k - 2) = 0
\]
\[
k^2 + 2k + 1 + 4k + 8 = 0
\]
\[
k^2 + 6k + 9 = 0
\]
\[
(k + 3)^2 = 0 \Rightarrow k = -3
\]
1 b(ii)

Roots of \( f(x) = 0 \)

\[
x = \frac{-(k + 1) \pm \sqrt{(k + 1)^2 - 4(1)(-k - 2)}}{2}
= \frac{-(k + 1) \pm \sqrt{k^2 + 6k + 9}}{2}
= \frac{-(k + 1) \pm (k + 3)}{2}
\]

\[
x = \frac{-k - 1 + k + 3}{2} \quad \text{or} \quad x = \frac{-k - 1 - k - 3}{2}
\]

\[
x = \frac{2}{2} \quad \text{or} \quad x = \frac{-2k - 4}{2}
\]

\[
x = 1 \quad \text{or} \quad x = -k - 2
\]

or

1 b(ii)

\[
f(x) = (x - 1)(x + k + 2) = 0
\]

\[
x - 1 = 0 \quad \text{or} \quad x + k + 2 = 0
\]

\[
x = 1 \quad \text{or} \quad x = -k - 2
\]

1b(iii)

Positive roots : \((-k - 2) > 0\)

\[-2 > k\]

Blunders (-3)
B1 Equal roots condition
B2 Expansion of \((k + 1)^2\) once only
B3 Indices
B4 Factors
B5 Roots formula once only
B6 Inequality sign
B7 Deduction of root from factor or no root
B8 Range

Slips (-1)
S1 Numerical

Attempts
A1 Equation not quadratic in b(i) gives \(A \neq 2\) at most in finish
A2 Using remainder theorem
1 (c) \( x + p \) is a factor of both \( ax^2 + b \) and \( ax^2 + bx - ac \).

(i) Show that \( p^2 = -\frac{b}{a} \) and that \( p = \frac{-b - ac}{b} \).

(ii) Hence show that \( p^2 + p^3 = c \).

\[ (i) \text{ Show } \quad p^2 = \frac{-b}{a}, \quad p = \frac{-b - ac}{b} \]

\[ (ii) \text{ Hence show that } \quad p^2 + p^3 = c \]

\[ (1) \text{ Show } \quad 5 \text{ marks} \]

\[ (ii) \text{ 10 marks} \]

1 (c) (i) 

\( (x + p) \) factor of \( ax^2 + b \) \( \Rightarrow f(-p) = 0 \)

\[ a(-p)^2 + b = 0 \]

\[ ap^2 = -b \]

\[ p^2 = -\frac{b}{a} \]

\( (x + p) \) factor of \( ax^2 + bx - ac \) \( \Rightarrow f(-p) = 0 \)

\[ a(-p)^2 + b(-p) - ac = 0 \]

\[ ap^2 - bp - ac = 0 \]

But \( ap^2 = -b \) from above \( \Rightarrow -b - ac = bp \)

\[ \frac{-b - ac}{b} = p \]

or

1 (c) (i) 

\[ \frac{ax - ap}{x + p} \]

\[ \frac{ax^2 + b}{ax^2 + bx - ac} \]

\[ \frac{apx}{-axp + b} \]

\[ \frac{-apx + b}{-apx - ap^2} \]

\[ \frac{ap^2 + b}{ap^2 + b} \]

Since \( (x - p) \) factor \( \Rightarrow ap^2 + b = 0 \)

\[ ap^2 = -b \]

\[ p^2 = \frac{-b}{a} \]

Since \( (x + p) \) factor

\[ -pb + ap^2 - ac = 0 \]

\[ -pb - b - ac = 0 \]

\[ -b - ac = pb \]

\[ \frac{-b - ac}{b} = p \]
\[ p^2 + p^3 = p^2(1 + p) \]
\[ = \frac{-b}{a} \left( 1 + \frac{-b - ac}{b} \right) \]
\[ = \frac{-b}{a} \left( \frac{b - b - ac}{b} \right) \]
\[ = \frac{bac}{ab} \]
\[ = c \]

Blunders (-3)
B1 Deduction root from factor
B2 Indices
B3 Not in required form

Slips (-1)
S1 Not changing sign when subtracting in division.
QUESTION 2

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<td>Att (2, 2, 2, 2)</td>
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**Part (a) 10 marks Att 3**

2 (a) Without using a calculator, solve the simultaneous equations

\[ \begin{align*}
  x + y + z &= 2 \\
 2x + y + z &= 3 \\
 x - 2y + 2z &= 15
\end{align*} \]

\[ \begin{align*}
   & (i) \quad x + y + z = 2 \\
   & (ii) \quad 2x + y + z = 3 \\
   & (iii) \quad x - 2y + 2z = 15
\end{align*} \]

\[ \begin{align*}
   & (i) \quad x + y + z = 2 \\
   & \quad \quad x = 1 \\
   & (ii) \quad 2x + y + z = 3 \\
   & \quad \quad y + z = 1 \\
   & \quad \quad -2y + 2z = 14 \\
   & \quad \quad -y + z = 7 \\
   & (iii) \quad 1 - 2y + 2z = 15 \\
\end{align*} \]

\[ \begin{align*}
   & (i) \quad y + z = 1 \\
   & \quad \quad y + 4 = 1 \\
   & \quad \quad y = -3 \\
   & (ii) \quad -y + z = 7 \\
   & \quad \quad z = 4 \\
   & (iii) \quad 2z = 8 \\
\end{align*} \]

\[ \begin{align*}
   & x = 1 \\
   & y = -3 \\
   & z = 4
\end{align*} \]

**Blunders (-3)**

B1 Multiplying one side of equation only
B2 Not finding 2nd unknown (having found 1st)
B3 Not finding 3rd unknown (having found 1st and 2nd)

**Slips (-1)**

S1 Numerical
S2 Not changing sign when subtracting

**Worthless**

W1 Trial and error only
2 (b) \( \alpha \) and \( \beta \) are the roots of the equation \( x^2 - 4x + 6 = 0 \).

(i) Find the value of \( \frac{1}{\alpha} + \frac{1}{\beta} \).

(ii) Find the quadratic equation whose roots are \( \frac{1}{\alpha} \) and \( \frac{1}{\beta} \).

Values of \( (\alpha + \beta) \) & \( \alpha \beta \) or solve quadr. 5 marks Att 2
Finish 5 marks Att 2
Correct Statement 5 marks Att 2
Finish 5 marks Att 2

2 (b) (i)
\[
x^2 - (4)x + (6) = 0
\]
\[
x^2 - (\alpha + \beta)x + (\alpha\beta) = 0
\]
\[
\therefore \quad \alpha + \beta = 4 \quad \alpha\beta = 6
\]
\[
\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{4}{6} = \frac{2}{3}
\]

2 (b) (ii)
\[
x^2 - (\text{sum of roots})x + (\text{product roots}) = 0
\]
\[
x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \left(\frac{1}{\alpha} \cdot \frac{1}{\beta}\right) = 0
\]
\[
x^2 - \left(\frac{2}{3}\right)x + \left(\frac{1}{6}\right) = 0
\]
\[
6x^2 - 4x + 1 = 0
\]

Blunders (-3)
B1 Indices
B2 Incorrect sum
B3 Incorrect product
B4 Statement incorrect

Slips (-1)
S1 Numerical
S2 Not as equation

Attempts
A1 Not quadratic equation
2 (c) (i) Prove that \( x + \frac{9}{x+2} \geq 4 \), where \( x + 2 > 0 \).

(ii) Prove that \( x + \frac{9}{x+a} \geq 6 - a \), where \( x + a > 0 \).

(i) Quadratic Inequality

\[
\begin{align*}
x + \frac{9}{x+2} & \geq 4 \\
\Leftrightarrow x(x+2) + 9 & \geq 4(x+2) \quad \text{[given } x + 2 > 0 \text{]} \\
\Leftrightarrow x^2 + 2x + 9 & \geq 4x + 8 \\
\Leftrightarrow x^2 - 2x + 1 & \geq 0 \\
\Leftrightarrow (x-1)^2 & \geq 0 \quad \text{True}
\end{align*}
\]

or

\[
\begin{align*}
x + \frac{9}{x+2} & = \frac{x(x+2) + 9 - 4(x+2)}{x+2} \\
& = \frac{x^2 + 2x + 9 - 4x - 8}{x+2} \\
& = \frac{x^2 - 2x + 1}{x+2} \\
& = \frac{(x-1)^2}{x+2} \geq 0 \quad \text{[given } x + 2 > 0 \text{]} 
\end{align*}
\]

(ii) Quadratic Inequality

\[
\begin{align*}
x + \frac{9}{x+a} & \geq 6 - a \\
\Leftrightarrow x(x+a) + 9 & \geq (6-a)(x+a) \quad \text{[given } x+a > 0 \text{]} \\
\Leftrightarrow x^2 + ax + 9 & \geq 6x - ax + 6a - a^2 \\
\Leftrightarrow x^2 + 2ax - 6x + a^2 - 6a + 9 & \geq 0 \\
\Leftrightarrow x^2 + 2(a-3)x + (a-3)^2 & \geq 0 \\
\text{Let } y = a - 3 \\
\Leftrightarrow x^2 + 2y x + y^2 & \geq 0 \\
\Leftrightarrow (x+y)^2 & \geq 0 \\
\Leftrightarrow (x+a-3)^2 & \geq 0 \quad \text{which is true, given } x+a > 0.
\end{align*}
\]
Blunders (-3)
B1 Inequality sign
B2 Factors
B3 Incorrect deduction or no deduction

Attempts
A1 Multiplication by \((x + 2)^2\)
A2 Multiplication by \((x + a)^2\)

Worthless
W1 Squares both sides
QUESTION 3

Part (a) 10 (5, 5) marks Att (2, 2)
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

Part (a) 10 (5, 5) marks Att 2

3

(a) Let \( A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix} \). Find \( A^2 - 2A \).

\[
A^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix}
\]

\[
2A = 2 \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix}
\]

\[
A^2 - 2A = \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

Slips
S1 Incorrect element
S2 Numerical
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

3 (b) Let \( z = -1 + i \), where \( i^2 = -1 \).

(i) Use De Moivre’s theorem to evaluate \( z^5 \) and \( z^9 \).

(ii) Show that \( z^5 + z^9 = 12z \).

(i) \( z \) in Polar Form

\[ z = -1 + i = r(\cos \theta + i \sin \theta) \]

\[ r = \sqrt{1^2 + 1^2} = \sqrt{2} \]

\[ \tan \alpha = \frac{1}{1} = 1 \]

\[ \therefore \alpha = \frac{\pi}{4}, \quad \theta = \frac{3\pi}{4} \]

\[ z^5 = \left[ r(\cos \theta + i \sin \theta) \right]^5 \]

\[ = 2^5 \left( \cos \frac{15\pi}{4} + i \sin \frac{15\pi}{4} \right) \]

\[ = 2^5 \left( \cos \frac{\pi}{4} + i \sin \frac{3\pi}{4} \right) \]

\[ = 2^5 \left( \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) \]

\[ = 2^5 (1 - i) = 4 - 4i \]

\[ z^9 = \left[ 2^5 \left( \cos \frac{\pi}{4} + i \sin \frac{3\pi}{4} \right) \right]^3 \]

\[ = 2^5 \left( \cos \frac{15\pi}{4} + i \sin \frac{15\pi}{4} \right) \]

\[ = 2^5 \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \]

\[ = 2^5 \left( -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \]

\[ = 2^4 (1 - i) = -16 + 16i \]

3b(ii) \( z^5 + z^9 = (4 - 4i) + (-16 + 16i) \)

\[ = 4 - 4i - 16 + 16i \]

\[ = -12 + 12i \]

\[ = 12(-1 + i) \]

\[ = 12z \]

Blunders (-3)

B1 Argument

B2 Modulus

B3 Trig definition

B4 Indices

B5 \( i \)

B6 Statement De Moivre once only

B7 Application De Moivre

Slips (-1)

S1 Trig value
3(c) (i) Find the two complex numbers \( a + bi \) for which \((a + bi)^2 = 15 + 8i\).

(ii) Solve the equation \( iz^2 + (2 - 3i)z + (-5 + 5i) = 0 \).

(i) Quadratic Equation 5 marks
Complex numbers 5 marks

\[
(a + bi)^2 = 15 + 8i
\]
\[
a^2 + 2abi + b^2i^2 = (15) + (8)i
\]
\[
(a^2 - b^2) + (2ab)i = (15) + (8)i
\]

(i): \( a^2 - b^2 = 15 \)

(ii): \( 2ab = 8 \)

(ii): \( 2ab = 8 \) \( \Rightarrow \) \( ab = 4 \) \( \Rightarrow b = \frac{4}{a} \)

Substitute into (i): \( \Rightarrow a^2 - \left(\frac{4}{a}\right)^2 = 15 \)
\[
a^2 - \frac{16}{a^2} = 15
\]

Let \( y = a^2 \), (so \( y > 0 \))
\[
\therefore \quad y - \frac{16}{y} = 15
\]
\[
y^2 - 16 = 15y
\]
\[
y^2 - 15y - 16 = 0
\]
\[
(y - 16)(y + 1) = 0
\]
\[
y = 16 \quad \text{or} \quad y = -1
\]
\[
y = a^2 \neq -1
\]
\[
\therefore a^2 = 16
\]
\[
a = \pm 4
\]

\[
a = 4: \quad b = \frac{4}{4} = 1 \Rightarrow 4 + i = z_1
\]
\[
a = -4: \quad b = \frac{-4}{4} = -1 \Rightarrow -4 - i = z_2
\]

**Blunders (-3)**
B1 Expansion \( (a + ib)^2 \)
B2 Indices
B3 \( i \)
B4 Not like to like
B5 Factors
B6 Quadratic formula
B7 Excess values (not real)
B8 Only one complex number found
B9 Incorrect deduction of root from factor
(ii) Using square root value
Completion

3 (c) (ii)

\[ iz^2 + (2 - 3i)z + (-5 + 5i) = 0 \]
\[ a = i ; \quad b = (2 - 3i) ; \quad c = (-5 + 5i) \]

\[
z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{-(2 - 3i) \pm \sqrt{(2 - 3i)^2 - 4(i)(-5 + 5i)}}{2i}
\]
\[
= \frac{-2 + 3i \pm \sqrt{4 - 12i + 9i^2 + 20i - 20i^2}}{2i}
\]
\[
= \frac{-2 + 3i \pm \sqrt{15 + 8i}}{2i}
\]
\[
= \frac{-2 + 3i \pm (4 + i)}{2i}
\]

\[
z_1 = \frac{-2 + 3i + (4 + i)}{2i}
\]
\[
z_1 = \frac{2 + 4i}{i} \cdot \frac{i}{i}
\]
\[
z_1 = \frac{1 + 2i}{i} \cdot \frac{i}{i}
\]
\[
z_1 = \frac{i - 2}{-1}
\]
\[
z_1 = 2 - i
\]

\[
z_2 = \frac{-2 + 3i - (4 + i)}{2i}
\]
\[
z_2 = \frac{-6 + 2i}{2i}
\]
\[
z_2 = \frac{-3 + i}{i} \cdot \frac{i}{i}
\]
\[
z_2 = \frac{-3i - 1}{-1}
\]
\[
z_2 = 1 + 3i
\]

**Blunders (-3)**
B1 Indices
B2 \( i \)
B3 Expansion of \((2 - 3i)^2\) once only
B4 Root formula once only
B5 \( i \) in denominator

**Slips (-1)**
S1 Numerical
**QUESTION 4**

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<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
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<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
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**Part (a) 10 (5, 5) marks Att (2, 2)**

4 (a) 
Show that \( \left( \begin{array}{c} n \\ 1 \end{array} \right) + \left( \begin{array}{c} n \\ 2 \end{array} \right) = \left( \begin{array}{c} n+1 \\ 2 \end{array} \right) \) for all natural numbers \( n \geq 2 \).

(a) L.H.S. 5 marks Att 2
R.H.S. 5 marks Att 2

4 (a)

L.H.S. = \( \left( \begin{array}{c} n \\ 1 \end{array} \right) + \left( \begin{array}{c} n \\ 2 \end{array} \right) = n + \frac{n(n-1)}{2} \)
\[ = \frac{2n + n^2 - n}{2} \]
\[ = \frac{n^2 + n}{2} \]

R.H.S. = \( \left( \begin{array}{c} n+1 \\ 2 \end{array} \right) = \frac{(n+1)n}{2} = \frac{n^2 + n}{2} \)

\[ \therefore \left( \begin{array}{c} n \\ 1 \end{array} \right) + \left( \begin{array}{c} n \\ 2 \end{array} \right) = \left( \begin{array}{c} n+1 \\ 2 \end{array} \right) \]

**Blunders (-3)**
B1 Indices
B2 Value \( \left( \begin{array}{c} n \\ r \end{array} \right) \)

**Attempts**
A1 Correct by using values for \( n \).
4 (b) \( u_1 = 5 \) and \( u_{n+1} = \frac{n}{n+1} u_n \) for all \( n \geq 1, x \in \mathbb{N} \).

(i) Write down the values of \( u_2, u_3, \) and \( u_4 \).

(ii) Hence, by inspection, write an expression for \( u_n \) in terms of \( n \).

(iii) Use induction to justify your answer for part (ii).

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<th>5 marks</th>
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<td>(ii) ( u_n )</td>
<td>5 marks</td>
<td>Att 2</td>
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<td>(iii) P(1) and P(( k ))</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
<tr>
<td>P(( k+1 ))</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
</tbody>
</table>

4 (b) 

(i) 
\[
\begin{align*}
  u_1 &= 5 = \frac{5}{1} \\
  u_2 &= \frac{1}{1+1} (5) = \frac{5}{2} \\
  u_3 &= \frac{2}{2+1} \left( \frac{5}{2} \right) = \frac{5}{3} \\
  u_4 &= \frac{3}{3+1} \left( \frac{5}{3} \right) = \frac{5}{4}
\end{align*}
\]

(ii) 
\[
  u_n = \frac{5}{n}
\]

(iii) To prove: \( u_n = \frac{5}{n} \)

P(1): \( n = 1 \): \( u_1 = \frac{5}{1} \Rightarrow \) true for \( n = 1 \)

Assume P(\( k \)): i.e., assume true for \( n = k \) \( \Rightarrow u_k = \frac{5}{k} \).

Deduce P(\( k+1 \)): i.e., prove true for \( n = k + 1 \):
\[
  u_{k+1} = \frac{k}{k+1} (u_k) = \frac{k}{k+1} \left( \frac{5}{k} \right) = \frac{5}{k+1}
\]

\( \therefore \) truth of P(\( k \)) implies truth of P(\( k+1 \)), and P(1) is true. 
\( \therefore \) true for all \( n \geq 1 \).

* Note: Accept P(1) as given

Blunders (-3)
B1 Incorrect term once only
B2 Incorrect deduction
B3 Incorrect P(\( k \))
B4 Incorrect P(\( k+1 \))
The sum of the first \( n \) terms of a series is given by \( S_n = n^2 \log_2 3 \).

(i) Find the \( n^{th} \) term and prove that the series is arithmetic.

(ii) How many of the terms of the series are less than \( 27 \log_2 12 \)?

\[ S_n = n^2 \log_2 3 \]
\[ S_{n-1} = (n-1)^2 \log_2 3 \]
\[ u_n = S_n - S_{n-1} = n^2 \log_2 3 - (n-1)^2 \log_2 3 \]
\[ = (\log_2 3)[n^2 - (n^2 - 2n + 1)] \]
\[ = (\log_2 3)[2n - 1] \]
\[ u_n = (2n - 1) \log_2 3 \]
\[ u_{n+1} = [2(n+1) - 1] \log_2 3 = (2n + 1) \log_2 3 \]
\[ d = u_{n+1} - u_n = (2n + 1) \log_2 3 - (2n - 1) \log_2 3 \]
\[ = \log_2 3 [2n + 1 - 2n + 1] \]
\[ = 2 \log_2 3 \text{ constant} \]

\[ \therefore \text{arithmetic, with } d = 2 \log_2 3 \]

\[ 12 \log_2 27 = 12 \log_2 (3^3) = 12[3 \log_2 3] = 36 \log_2 3 \]

Let \( (2n - 1) \log_2 3 \leq 36 \log_2 3 \)
\[ 2n - 1 \leq 36 \]
\[ 2n \leq 37 \]
\[ n \leq 18 \frac{1}{2} \]

So the first 18 terms are less than \( 12 \log_2 27 \).

Blunders (-3)
B1 AP formula once only
B2 Incorrect terms (must be consecutive)
B3 Log laws
B4 Indices
B5 Incorrect \( \log_2 27 \) or no \( \log_2 27 \)
B6 Inequality sign
B7 Incorrect value or no value
B8 \( U_n = S_{n+1} - S_n \)

Slips (-1)
S1 Numerical
QUESTION 5

Part (a) 10 (5, 5) marks  
Part (b) 20 (5, 5, 5, 5) marks  
Part (c) 20 (5, 5, 5, 5) marks

5 (a) Plot, on the number line, the values of \( x \) that satisfy the inequality \( |x+1| \leq 2 \), where \( x \in \mathbb{Z} \).

\[
\begin{align*}
\text{Inequality} & \quad \text{5 marks} & \text{Solution set plotted} & \quad \text{5 marks} \\
5 (a) & \quad |x+1| \leq 2 \quad \Rightarrow \quad -2 \leq x + 1 \leq 2. \\
& \quad \therefore -3 \leq x \leq 1.
\end{align*}
\]

or

\[
\begin{align*}
5(a) & \quad |x+1| \leq 2 \\
& \quad (x+1)^2 \leq (2)^2 \\
& \quad x^2 + 2x + 1 \leq 4 \\
& \quad x^2 + 2x - 3 \leq 0 \\
& \quad \text{Graph: } y = x^2 + 2x - 3 \\
& \quad \text{roots: } (x+3)(x-1) = 0 \\
& \quad x = -3, x = 1 \\
& \quad \therefore -3 \leq x \leq 1.
\end{align*}
\]

Blunders (-3)
- B1 Upper limit
- B2 Lower limit
- B3 Expansion \((x+1)^2\) once only
- B4 Inequality sign
- B5 Indices
- B6 Factors once only
- B7 Root formula once only
- B8 Deduction root from factor
- B9 Incorrect range
- B10 Set not plotted

Slips (-1)
- S1 Numerical

Attempts
- A1 One inequality sign
- A2 Inequality signs ignored
- A3 Scaled and numbered line
5 (b) In the expansion of \((2x - \frac{1}{x^2})^9\),

(i) find the general term.

(ii) find the value of the term independent of \(x\).

**General Term**

<table>
<thead>
<tr>
<th>Marking</th>
<th>5 marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Att</td>
<td>2</td>
</tr>
</tbody>
</table>

**Power of \(x\)**

<table>
<thead>
<tr>
<th>Marking</th>
<th>5 marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Att</td>
<td>2</td>
</tr>
</tbody>
</table>

**Value**

<table>
<thead>
<tr>
<th>Marking</th>
<th>5 marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Att</td>
<td>2</td>
</tr>
</tbody>
</table>

5 (b) \(2x + \left(-\frac{1}{x^2}\right)^b\)

(i) General Term: \(u_{r+1} = \binom{9}{r} (2x)^{9-r} \left(-\frac{1}{x^2}\right)^r\) \[= \binom{9}{r} x^{9-r} \left(-\frac{1}{x^2}\right)^r \]
\[= k x^{9-3r} \]

(ii) Term independent or \(x\) is the term with \(x^0\): \[9 - 3r = 0 \Rightarrow r = 3\]
\[u_4 = \frac{9}{3} (2x)^6 \left(-\frac{1}{x^2}\right)^3 = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} (64x^6) \left(-\frac{1}{x^6}\right) \]
\[= -5376\]

or

\[\frac{9}{3} (2x)^6 \left(-\frac{1}{x^2}\right)^3 = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} (2^6 x^6) (-1)^3 \frac{1}{x^6} \]
\[= -5376\]

**Blunders (-3)**

B1 General term
B2 Error Binomial expansion once only
B3 Indices
B4 Value \(\binom{n}{r}\) or no value \(\binom{n}{r}\)
B5 \(x^x \neq 1\)
B6 Correct term in expansion not identified.

**Slips (-1)**

S1 Numerical
5 (c) The $n^{th}$ term of a series is given by $nx^n$, where $|x| < 1$.

(i) Find an expression for $S_n$, the sum to $n$ terms of the series.

(ii) Hence, find the sum to infinity of the series.

5 (c) (i)

\[ s_n = x + 2x^2 + 3x^3 + \ldots + (n-1)x^{n-1} + nx^n \]

\[ xs_n = x^2 + 2x^3 + 3x^4 + \ldots + (n-1)x^n + nx^{n+1} \]

\[ s_n - xs_n = x + x^2 + x^3 + \ldots + x^n + nx^{n+1} - nx^{n+1} \]

\[ = \left[ x + x^2 + x^3 + \ldots + x^n \right] - nx^{n+1} \]

\[ s_n = \frac{a(1-r^n)}{1-r} - nx^{n+1} \]

\[ s_n = \frac{a(1-r^n)}{1-r} - nx^{n+1} \]

5 (c) (ii)

$|x| < 1$. Hence, as $n \to \infty$, $x^n \to 0$.

\[ S_\infty = \frac{x(1-0)}{1-x^2} - 0 \]

\[ S_\infty = \frac{x}{1-x^2} \]

Blunders (-3)

B1 Indices
B2 GP formula
B3 Incorrect ‘a’
B4 Incorrect ‘r’
B5 $S_n$ not isolated
B6 $x^n \to 0$ in (ii)

Slips (-1)

S1 Not changing sign when subtracting
**QUESTION 6**

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

**Part (a) 10 marks Att 3**

6 (a) Differentiate \( \frac{x^2 - 1}{x^2 + 1} \) with respect to \( x \).

(a) 10 marks Att 3

\[
y = \frac{x^2 - 1}{x^2 + 1}
\]

\[
\frac{dy}{dx} = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}
\]

**Blunders (-3)**
B1 Indices
B2 Differentiation

**Attempts**
A1 Error in differentiation formula

**Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)**

6 (b) (i) Differentiate \( \frac{1}{x} \) with respect to \( x \) from first principles.

(ii) Find the equation of the tangent to \( y = \frac{1}{x} \) at the point \( (2, \frac{1}{2}) \).

(i) \( f(x + h) - f(x) \) simplified 5 marks Att 2

Finish 5 marks Att 2

6 (b) (i)

\[
f(x) = \frac{1}{x}, \quad f(x + h) = \frac{1}{x + h}
\]

\[
f(x + h) - f(x) = \frac{1}{x + h} - \frac{1}{x} = \frac{x - (x + h)}{x(x + h)}
\]

\[
f(x + h) - f(x) = -\frac{h}{x(x + h)}
\]

\[
f(x + h) - f(x) = -\frac{1}{x(x + h)}
\]

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = -\frac{1}{x^2}
\]
or

6(b)(i)

\[
y = \frac{1}{x}
\]

\[
y + \Delta y = \frac{1}{x + \Delta x}
\]

\[
\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = \frac{-\Delta x}{x(x + \Delta x)}
\]

\[
\frac{\Delta y}{\Delta x} = \frac{-1}{x(x + \Delta x)}
\]

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{-1}{x^2}
\]

**Blunders (-3)**

B1 \( f(x + h) \)

B2 Indices

B3 No limits shown or implied or no indication \( \to 0 \)

B4 \( h \to \infty \)

**Worthless**

W1 Not 1st principles

(ii) Slope 5 marks Att 2

Equation 5 marks Att 2

6 (b) (ii) \[
\frac{dy}{dx} = \frac{-1}{x^2}
\]

At \( \left( 2, \frac{1}{2} \right) \)

\[
\text{slope} = \frac{dy}{dx} = \frac{-1}{(2)^2} = \frac{-1}{4}
\]

\[
\therefore \text{Tangent is line through } \left( 2, \frac{1}{2} \right) \text{ with slope } m = \frac{-1}{4}
\]

\[
y - y_1 = m(x - x_1)
\]

\[
y - \frac{1}{2} = \frac{-1}{4}(x - 2)
\]

\[
4y - 2 = -x + 2
\]

\[
x + 4y = 4
\]

**Blunders (-3)**

B1 Differentiation

B2 Indices

B3 Equation formula line

B4 Substituting values into formula once only

**Slips (-1)**

S1 Numerical
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

6 (c) Let \( f(x) = \tan^{-1} \left( \frac{x}{2} \right) \) and \( g(x) = \tan^{-1} \left( \frac{2}{x} \right) \), for \( x > 0 \).

(i) Find \( f'(x) \) and \( g'(x) \).

(ii) Hence, show that \( f(x) + g(x) \) is constant.

(iii) Find the value of \( f(x) + g(x) \).

(i) \( f'(x) \) 5 marks Att 2

\[ f'(x) = \frac{2}{4 + x^2} \]

\( g(x) = \tan^{-1} \left( \frac{2}{x} \right) \)

\( g'(x) = \frac{1}{1 + \left( \frac{2}{x} \right)^2} \cdot \left( -\frac{2}{x^2} \right) = \frac{-2}{4 + x^2} \)

6(c)(ii) \( f'(x) + g'(x) = \frac{2}{4 + x^2} + \frac{-2}{4 + x^2} = 0 \).

Derivative of \( f(x) + g(x) \) is 0, so \( f(x) + g(x) \) is constant.

or

6(c)(ii) \( f'(x) + g'(x) = 0 \Rightarrow \int [f'(x) + g'(x)] dx = k \)

\( \Rightarrow f(x) + g(x) = k \) (constant)

6(c)(iii) Let \( \tan^{-1} \left( \frac{x}{2} \right) + \tan^{-1} \left( \frac{2}{x} \right) = k \) \( [x > 0] \)

Let \( x = 2: \tan^{-1} (1) + \tan^{-1} (1) = k \)

\[ \frac{\pi}{4} + \frac{\pi}{4} = k \]

\[ k = \frac{\pi}{2} \]

* Note: Any value of \( x \) in the domain can be used in place of \( x = 2 \) above.

or
\[ \alpha + \beta = \frac{\pi}{2} \]

\[ \tan \alpha = \frac{x}{2} \Rightarrow \alpha = \tan^{-1}\left(\frac{x}{2}\right) \]

\[ \tan \beta = \frac{2}{x} \Rightarrow \beta = \tan^{-1}\left(\frac{2}{x}\right) \]

\[ \tan^{-1}\left(\frac{x}{2}\right) + \tan^{-1}\left(\frac{2}{x}\right) = \alpha + \beta = \frac{\pi}{2} \]
**QUESTION 7**

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

### Part (a) 10 marks Att 3

#### 7 (a)
Taking 1 as a first approximation of a root of \( x^3 + 2x - 4 = 0 \), use the Newton Raphson method to calculate a second approximation of this root.

#### (a) 10 marks Att 3

\[
\begin{align*}
7 \text{ (a)} & \quad f(x) = x^3 + 2x - 4 \\
& \quad f'(x) = 3x^2 + 2 \\
& \quad f(1) = (1)^3 + 2(1) - 4 = -1 \\
& \quad f'(1) = 3(1)^2 + 2 = 5 \\
\end{align*}
\]

\[
\begin{align*}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
x_1 &= 1 \\
= 1 - \left( \frac{-1}{5} \right) &= 1 + \frac{1}{5} = \frac{6}{5} \\
\end{align*}
\]

*Blunder (-3)*
B1 Newton-Raphson formula once only
B2 Differentiation
B3 Indices
B4 \( x_i \neq 1 \) once only

*Slips (-1)*
S1 Numerical
S2 Answer not tidied up

### Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

#### 7 (b) (i) Find the equation of the tangent to the curve \( 3x^2 + y^2 = 28 \) at the point \( (2, -4) \).

#### (ii) \( x = e^t \cos t \) and \( y = e^t \sin t \). Show that \( \frac{dy}{dx} = \frac{x + y}{x - y} \).

#### (i) Differentiation 5 marks Att 2

<table>
<thead>
<tr>
<th>Equation</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 (b)(i)</td>
<td>( 3x^2 + y^2 = 28 )</td>
<td></td>
</tr>
<tr>
<td>&amp; ( 6x + 2y \frac{dy}{dx} = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; ( 2y \left( \frac{dy}{dx} \right) = -6x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; ( \frac{dy}{dx} = -6x = -3x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; ( \frac{dy}{dx} = \frac{-6x}{2y} = \frac{-3x}{y} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
At (2, -4), slope \( \frac{dy}{dx} = \frac{-3(2)}{-4} = \frac{3}{2} \)

Tangent is line through (2, -4) with slope \( m = \frac{3}{2} \)

\[
(y - y_1) = m(x - x_1)
\]

\[
y - (-4) = \frac{3}{2}(x - 2)
\]

\[
2(y + 4) = 3(x - 2)
\]

\[
2y + 8 = 3x - 6
\]

\[
3x - 2y - 14 = 0
\]

**Blunders (-3)**

B1 Differentiation

B2 Incorrect values or no values

B3 Indices

B4 Equation of tangent

B5 Substituting values into formula once only

**Slips (-1)**

S1 Numerical

**Worthless**

W1 Integration

W2 No differentiation in 1st 5 marks

(ii) \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \)

5 marks \hspace{1cm} \text{Att 2}

\[
\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{e^t \cos t + e^t \sin t}{e^t \cos t - e^t \sin t} = \frac{x + y}{x - y}
\]

*Note: oversimplified differentiation in first 5 marks leads to Att 2 at most in second marks*

**Blunders (-3)**

B1 Differentiation

B2 Indices

B3 Incorrect \( \frac{dy}{dx} \)

B4 Answer not in required form

**Attempts**

A1 Blunder in differentiation formula

**Worthless**

W1 Integration
7 (c) \( f(x) = \log_e 3x - 3x \), where \( x > 0 \).

(i) Show that \( \left( \frac{1}{3}, -1 \right) \) is a local maximum point of \( f(x) \).

(ii) Deduce that the graph of \( f(x) \) does not intersect the \( x \)-axis.

(i) Differentiation 5 marks Att 2
Max Value 5 marks Att 2
(ii) Only one root for \( f'(x) = 0 \) 5 marks Att 2

Absolute max pt. 5 marks Att 2

7 (c)(i) \( f(x) = \ln(3x) - 3x \quad x > 0 \)
\[ f'(x) = \frac{1}{3x} (3) - 3 = \frac{1}{x} - 3. \]
\[ f''(x) = -\frac{1}{x^2}. \]
Local max/min: \( f'(x) = 0 \Rightarrow \frac{1}{x} - 3 = 0 \Rightarrow \frac{1}{x} = 3 \Rightarrow x = \frac{1}{3}. \)
\[ x = \frac{1}{3} \Rightarrow f''(x) = -\frac{1}{x^2} = -\frac{1}{(\frac{1}{3})^2} < 0 \Rightarrow \text{local max at } x = \frac{1}{3} \]
\[ x = \frac{1}{3} \Rightarrow f(x) = \ln(3x) - (3x) = \ln(1) - (1) = 0 - 1 = -1 \Rightarrow \text{Local max at } \left( \frac{1}{3}, -1 \right) \]

or

7(c)(i) \( f(x) = \ln 3x - 3x \)
\[ f'(x) = \frac{1}{x} - 3 \]
\[ x = \frac{1}{3} \Rightarrow f''(x) = \frac{1}{(\frac{1}{3})} - 3 = 3 - 3 = 0 \Rightarrow \text{turning pt at } x = \frac{1}{3}. \]
\[ f''(x) = -\frac{1}{x^2} < 0 \text{ for all } x \Rightarrow \text{local max pt at } x = \frac{1}{3} \]
\[ x = \frac{1}{3} \Rightarrow y = \ln(3x) - 3x = \ln(1) - 3 \left( \frac{1}{3} \right) = -1 \Rightarrow \text{local max is at } \left( \frac{1}{3}, -1 \right) \]
(c)(ii) \( f'(x) \) has only one root.
This implies that the local max. above is the only turning point.
And \( f(x) \) is continuous, so the local max pt above is an absolute max. point.

Since max pt \( \left( \frac{1}{3}, -1 \right) \) is below \( x \)-axis, the whole graph must lie below \( x \)-axis.

Thus, \( f(x) = 0 \) has no roots, since graph does not cut the \( x \)-axis.

* Accept work showing max point to be the only turning point and below \( x \)-axis, with or without a diagram.
* No need to mention “absolute” in answer.
* No need to mention continuity.

Blunders (-3)
B1 Differentiation
B2 Not testing in \( f''(x) \) for max
B3 Incorrect deduction or no deduction from test
B4 Incorrect \( y \) value or no \( y \) value
B5 Factors once only.

Slips (-1)
S1 \( \ln 1 \neq 0 \)

Worthless
W1 No differentiation
# QUESTION 8

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2)</td>
</tr>
</tbody>
</table>

### Part (a) 10 (5, 5) marks Att (2, 2)

8. (a) Find

(i) \( \int x^3 \, dx \)

(ii) \( \int \frac{1}{x^3} \, dx \).

<table>
<thead>
<tr>
<th>Q8 (a) (i)</th>
<th>5 marks</th>
<th>Att2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q8 (a) (ii)</td>
<td>5 marks</td>
<td>Att2</td>
</tr>
</tbody>
</table>

\[ \int x^3 \, dx = \frac{x^4}{4} + c \]

\[ \int \frac{1}{x^3} \, dx = \int x^{-3} \, dx = \frac{x^{-2}}{-2} + c = \frac{-1}{2x^2} + c \]

* If \( c \) shown once, then no penalty

**Blunders (-3)**

B1 Integration

B2 Indices

B3 No ‘c’ (penalize 1st integration)

**Attempts**

A1 Only ‘c’ correct

**Worthless**

W1 Differentiation instead of integration

### Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2)

8 (b) (i) Evaluate \( \int_0^4 x \sqrt{x^2 + 9} \, dx \).

(ii) \( f \) is a function such that \( f'(x) = 6 - \sin x \) and \( f \left( \frac{\pi}{3} \right) = 2\pi \). Find \( f(x) \).

<table>
<thead>
<tr>
<th>8 (b) (i)</th>
<th>Integration</th>
<th>Value</th>
<th>5 marks</th>
<th>Att2</th>
<th>5 marks</th>
<th>Att2</th>
</tr>
</thead>
</table>

\[ \int_0^4 x \sqrt{x^2 + 9} \, dx \]

\[ = \int \left( \sqrt{x^2 + 9} \right) x \, dx \]

\[ = \frac{1}{2} \int w^1 \, dw \]

\[ = \frac{1}{2} \left[ \frac{w^2}{2} \right] \]

\[ = \frac{1}{3} \left[ (x^2 + 9)^{3/2} \right]_0^4 \]

\[ = \frac{1}{3} \left[ (25)^{3/2} - (9)^{3/2} \right] = \frac{1}{3} [125 - 27] = \frac{98}{3} \]

* Incorrect substitution and unable to finish yields attempt at most.
Blunders (-3)
B1 Integration
B2 Indices
B3 Differentiation
B4 Limits
B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits

Slips (-1)
S1 Answer not “tidied up”.

Worthless
W1 Differentiation instead of integration except where other work merits attempt

(ii) $f(x)$

Value of $c$ | 5 marks | Att 2
---|---|---

8 (b) (ii) | $f''(x) = 6 - \sin x$ |

$f(x) = 6x + \cos x + c$

$f\left(\frac{\pi}{2}\right) = 6\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) + c = 2\pi$

$2\pi + \frac{\pi}{2} + c = 2\pi$

$c = -\frac{1}{2}$

$\Rightarrow f(x) = 6x + \cos x - \frac{1}{2}$

Blunders (-3)
B1 Integration
B2 No ‘c’

Slips (-1)
S1 Trig Value

Worthless
W1 Differentiation instead of integration, except where other work merits attempt.
The line \(2x - y - 10 = 0\) is a tangent to the curve \(y = x^2 - 9\), as shown. The shaded region is bounded by the line, the curve and the \(x\)-axis. Calculate the area of this region.

**Point (1, -8)** 5 marks  
**Points (3, 0) and (5, 0)** 5 marks  
**Area under curve between 1 and 3** 5 marks  
**Finish** 5 marks

\(x^2 - y = 9\)

\(y = 2x - 10\)

\(2x - 10 = x^2 - 9\)

\(0 = x^2 - 2x + 1\)

\(0 = (x - 1)^2\)

\(\Rightarrow x = 1\)

\(y = 2(1) - 10\)

\(y = -8\)

\(\Rightarrow t(1, -8)\) and \(a(1, 0)\)

Co-ords of \(b\):

\(y = x^2 - 9\)

\(y = 0 : x^2 - 9 = 0\)

\(x^2 = 9\)

\(x = \pm 3\)

\(b(3, 0)\)

Co-ords of \(c\):

\(y = 2x - 10\)

\(y = 0 : 0 = 2x - 10\)

\(x = 5\)

\(c(5, 0)\)

Shaded area = area \(\Delta act\) – area under curve

\(|ac| = 4\) and \(|at| = 8\) \(\Rightarrow\) area \(\Delta act = \frac{1}{2}|ac||at| = \frac{1}{2}(4)(8) = 16\)

or

Area \(\Delta act = \int_{1}^{3} y \, dx = \int_{1}^{3} (2x - 10) \, dx = \left[ x^2 - 10x \right]_{1}^{3} = (25 - 50) - (1 - 10) = (-25) - (-9) = 16.\)

Area under curve \(\int_{1}^{3} y \, dx = \int_{1}^{3} (x^2 - 9) \, dx = \left[ \frac{x^3}{3} - 9x \right]_{1}^{3} = (9 - 27) - \left( \frac{1}{3} - 9 \right) = \frac{28}{3}.\)

Shaded area = \(16 - \frac{28}{3} = \frac{48 - 28}{3} = \frac{20}{3}\)
Blunders (-3)
B1 Integration
B2 Indices
B3 Factors once only
B4 Calculation point of tangency
B5 Calculation of point where curve cuts x-axis
B6 Calculation of point where line cuts x-axis
B7 Error in area triangle
B8 Error in area formula
B9 Incorrect order in applying limits
B10 Not calculating substituted limits
B11 Error with line
B12 Error with curve
B13 Uses $\pi \int y \, dx$ for area formula

Attempts
A1 Uses volume formula
A2 Uses $y^2$ in formula

Worthless
W1 Differentiation instead of integration except where other work merits attempt
W2 Wrong area formula and no work
GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates’ work as follows:
   - Blunders – mathematical errors/omissions (-3)
   - Slips – numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any correct, relevant step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2,…etc.

4. The phrase “hit or miss” means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase “and stops” means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The same error in the same section of a question is penalised once only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5.50 may be written as €5,50.
1. (a) The following parametric equations define a circle:
\[ x = 5 + 7\cos\theta, \quad y = 7\sin\theta, \] where \( \theta \in \mathbb{R} \).

What is the Cartesian equation of the circle?

\[
\begin{align*}
(x - 5)^2 &= 49\cos^2\theta \quad \text{and} \quad y^2 = 49\sin^2\theta, \\
(x - 5)^2 + y^2 &= 49(\cos^2\theta + \sin^2\theta) \\
\therefore (x - 5)^2 + y^2 &= 49 \quad \text{or} \quad x^2 + y^2 - 10x - 24 = 0.
\end{align*}
\]

**Blunders**

B1 Error in transposition.
B2 Fails to square.
B3 \( \sin^2\theta + \cos^2\theta \neq 1 \).

**Slips**

S1 Arithmetic errors.

**Attempts**

A1 Writes down equation of a circle without any further work.
Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

<table>
<thead>
<tr>
<th>1 (b)</th>
<th>20 (5, 5, 5, 5) marks</th>
<th>Att (2, 2, 2, 2)</th>
</tr>
</thead>
</table>

\[ \begin{align*} x^2 + y^2 - 4x - 6y + 5 &= 0 \quad \text{and} \quad x^2 + y^2 - 6x - 8y + 23 &= 0 \end{align*} \]

are two circles.

(i) Prove that the circles touch internally.

(ii) Find the coordinates of the point of contact of the two circles.

| (i) One centre and radius (same circle) | 5 marks | Att 2 |
| Distance between centres | 5 marks | Att 2 |
| Conclusion | 5 marks | Att 2 |

1 (b) (i)

\[ \begin{align*} x^2 + y^2 - 4x - 6y + 5 &= 0 \quad \text{has centre} \quad c_1(2, 3) \quad \text{and radius} \quad r_1 = \sqrt{4 + 9 - 5} = \sqrt{8} = 2\sqrt{2}. \\
 x^2 + y^2 - 6x - 8y + 23 &= 0 \quad \text{has centre} \quad c_2(3, 4) \quad \text{and radius} \quad r_2 = \sqrt{9 + 16 - 23} = \sqrt{2}. \\
|c_1c_2| &= \sqrt{(2 - 3)^2 + (3 - 4)^2} = \sqrt{2}. \\
 r_1 - r_2 &= 2\sqrt{2} - \sqrt{2} = \sqrt{2} = |c_1c_2|. \quad \therefore \text{Circles touch internally.} \end{align*} \]

Blunders

B1 Error in finding centre or radius.
B2 Error in distance formula.
B3 Fails to show internal touching

Slips

S1 Arithmetic

1 (b) (ii)

\[ \begin{align*} x^2 + y^2 - 4x - 6y + 5 &= 0 \\
 x^2 + y^2 - 6x - 8y + 23 &= 0 \\
 2x + 2y - 18 &= 0 \quad \Rightarrow \quad x + y - 9 = 0 \quad \Rightarrow \quad x = 9 - y. \\
 (9 - y)^2 + y^2 - 4(9 - y) - 6y + 5 &= 0 \quad \Rightarrow \quad 81 - 18y + y^2 + y^2 - 36 + 4y - 6y + 5 = 0 \\
 2y^2 - 20y + 50 &= 0 \quad \Rightarrow \quad y^2 - 10y + 25 = 0 \quad \Rightarrow \quad (y - 5)^2 = 0. \quad \therefore y = 5, \quad x = 4. \quad \text{Point is} \quad (4, 5). \end{align*} \]

* Accept correct answer without work.

Blunders

B1 Error in finding equation of the radical axis (common tangent)
B2 Error in substitution
B3 Error in solving quadratic

Slips

S1 Arithmetic error.
A circle has its centre in the first quadrant.
The $x$-axis is a tangent to the circle at the point $(3, 0)$.
The circle cuts the $y$-axis at points that are 8 units apart.
Find the equation of the circle.

\[
\begin{align*}
\text{x-value of centre} & : 5 \text{ marks} \\
\text{ab is perp bisector} & : 5 \text{ marks} \\
\text{Radius} & : 5 \text{ marks} \\
\text{Equation} & : 5 \text{ marks}
\end{align*}
\]

\[
\begin{align*}
x \text{ value of centre is 3 as } x-\text{axis tangent at (3, 0).} \\
ab & \text{ is perpendicular bisector of chord, } \therefore \sqrt{bc} = 4. \\
\text{Triangle abc is right-angled } \Rightarrow & \quad ac = r = \sqrt{3^2 + 4^2} = 5. \\
|ac| = 5 = |ad| & \Rightarrow y \text{ value of centre is 5.} \\
\text{Circle has centre (3, 5) and radius length 5.} \\
\therefore & \quad \text{Circle: } (x - 3)^2 + (y - 5)^2 = 25 \text{ or } x^2 + y^2 - 6x - 10y + 9 = 0.
\end{align*}
\]

\textbf{Blunders}
\begin{itemize}
  \item B1 Incorrect $x$-coordinate of centre
  \item B2 $|bc| \neq 4$
  \item B3 Error in Pythagoras
  \item B4 Error in forming equation of the circle.
\end{itemize}

\textbf{Slips}
\begin{itemize}
  \item S1 Arithmetic
\end{itemize}
QUESTION 2

Part (a) 10 marks Att 3

Part (b) 20 (10, 10) marks Att (3, 3)

Part (c) 20 (10, 5, 5) marks Att (3, 2, 2)

Part (a) 10 marks Att 3

2. (a) \( \vec{x} = -2 \hat{i} + 5 \hat{j} \) and \( \vec{y} = -6 \hat{i} - 8 \hat{j} \). Express \( \vec{y} \) in terms of \( \hat{i} \) and \( \hat{j} \).

(a) 10 marks Att 3

2 (a) \( \vec{x} + \vec{y} = \vec{y} = -8 \hat{i} - 3 \hat{j} \).

Or \( \vec{xy} = -6 \hat{i} - 8 \hat{j} \Rightarrow \vec{y} - \vec{x} = -6 \hat{i} - 8 \hat{j} \). \(.\) \( \vec{y} = -6 \hat{i} - 8 \hat{j} - 2 \hat{i} + 5 \hat{j} = -8 \hat{i} - 3 \hat{j} \).

Blunders
B1 \( \vec{xy} \neq \vec{y} - \vec{x} \)
B2 Error in transposing

Slips
S1 Arithmetic

Part (b) 20 (10, 10) marks Att (3, 3)

(b) \( \vec{a} = 5 \hat{i} \) and \( \vec{b} = \sqrt{3} \hat{i} + 3 \hat{j} \).

(i) Show that \( \vec{ab} \) is not perpendicular to \( \vec{b} \).

(ii) Find the value of the real number \( k \), given that \( \vec{c} = k \vec{b} \) and \( \vec{ac} \perp \vec{b} \).

(b) (i) 10 marks Att 3

2 (b) (i)

\[ \vec{ab} = \vec{b} - \vec{a} = \sqrt{3} \hat{i} + 3 \hat{j} - 5 \hat{i} = (\sqrt{3} - 5) \hat{i} + 3 \hat{j}. \]

\[ \vec{ab} \cdot \vec{b} = \sqrt{3}(\sqrt{3} - 5) + 9 \neq 0. \text{ Not perpendicular.} \]

Blunders
B1 \( \vec{ab} \neq \vec{b} - \vec{a} \)
B2 Error in transposing
B3 No conclusion

Slips
S1 Arithmetic
(b) (ii) 10 marks Att 3

\[
\vec{c} = k \vec{b} = \sqrt{3}k \vec{i} + 3k \vec{j}, \quad \vec{ac} = \vec{c} - \vec{a} = (\sqrt{3}k - 5)\vec{i} + 3k \vec{j}.
\]

\[
\vec{ac} \perp \vec{b} \Rightarrow \vec{ac} \cdot \vec{b} = 0. \quad \therefore (\sqrt{3}k - 5)(\sqrt{3} + 9k) = 0 \Rightarrow 12k = 5\sqrt{3} \Rightarrow k = \frac{5\sqrt{3}}{12}.
\]

Blunders
B1 \(\vec{ac} \neq \vec{c} - \vec{a}\)
B2 Transposition errors
B3 No use of \(\vec{ac} \cdot \vec{b} = 0\)

Slips
S1 Arithmetic

Part (c) 20 (10, 5, 5) marks Att (3, 2, 2)

\[
\vec{p} = 3\vec{i} + 4\vec{j} \quad \text{and} \quad \vec{q} = 5\vec{i} + 12\vec{j}.
\]

\[
\vec{r} = \frac{65t}{16} \left( \frac{\vec{p}}{|\vec{p}|} + \frac{\vec{q}}{|\vec{q}|} \right), \quad \text{where} \quad t > 0.
\]

(i) Express \(\vec{r}\) in terms of \(\vec{i}\) and \(\vec{j}\).

(ii) Find \(\vec{p} \cdot \vec{r}\) and \(\vec{q} \cdot \vec{r}\).

(iii) Hence, show that \(r\) is on the bisector of \(\angle poq\), where \(o\) is the origin.

Part (c) (i) 10 marks Att 3

\[
\vec{r} = \frac{65t}{16} \left( \frac{3\vec{i} + 4\vec{j}}{5} + \frac{5\vec{i} + 12\vec{j}}{13} \right)
\]

\[
\vec{r} = \frac{65t}{16} \left( \frac{39\vec{i} + 52\vec{j} + 25\vec{i} + 60\vec{j}}{65} \right) = \frac{t}{16} \left( 64\vec{i} + 112\vec{j} \right) \quad \therefore \quad \vec{r} = t \left( 4\vec{i} + 7\vec{j} \right).
\]

Blunders
B1 Error in \(|\vec{p}|\) or \(|\vec{i}|\)
B2 Ignores \(t\) or \(t = \) some value.
S1 Arithmetic
Part (c) (ii)  

<table>
<thead>
<tr>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>
| 2 (c) (ii)  

\[ \overrightarrow{p} \cdot \overrightarrow{r} = \left( 3 \hat{i} + 4 \hat{j} \right) \left( 4t \hat{i} + 7t \hat{j} \right) = 12t + 28t = 40t. \]

\[ \overrightarrow{q} \cdot \overrightarrow{r} = \left( 5 \hat{i} + 12 \hat{j} \right) \left( 4t \hat{i} + 7t \hat{j} \right) = 20t + 84t = 104t. \]

**Blunders**  
B1 Error in calculating scalar product

**Slips**  
S1 Arithmetic

Part (c) (iii)  

<table>
<thead>
<tr>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>
| 2 (c) (iii)  

\[ \overrightarrow{p} \cdot \overrightarrow{r} = \left\| \overrightarrow{p} \right\| \left\| \overrightarrow{r} \right\| \cos \theta \Rightarrow 5t \sqrt{65} \cos \theta = 40t \Rightarrow \cos \theta = \frac{40}{5\sqrt{65}} = \frac{8}{\sqrt{65}}. \]

\[ \overrightarrow{q} \cdot \overrightarrow{r} = \left\| \overrightarrow{q} \right\| \left\| \overrightarrow{r} \right\| \cos \theta \Rightarrow 13t \sqrt{65} = 104t \Rightarrow \cos \theta = \frac{104}{13\sqrt{65}} = \frac{8}{\sqrt{65}}. \]

\[ \therefore \overrightarrow{r} \text{ is on bisector of } \angle \overrightarrow{p} \overrightarrow{q}. \]

**Blunders**  
B1 Error in scalar product formula  
B2 Error in modulus.

**Slips**  
S1 Arithmetic errors
### QUESTION 3

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 10) marks</td>
<td>Att (2, 2, 3)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (10, 5, 5) marks</td>
<td>Att (3, 2, 2)</td>
</tr>
</tbody>
</table>

3. (a) Find the area of the triangle with vertices $(1, 1), (8, -5)$ and $(5, -2)$.

(a) 10 marks

3 (a)

<table>
<thead>
<tr>
<th>(a)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
</table>

Area of triangle = $\frac{1}{2} |x_1y_2 - x_2y_1| = \frac{1}{2} | -21 + 24 | = \frac{3}{2}$.

or

Use $\frac{1}{2}$ base $\times$ perp. height:

<table>
<thead>
<tr>
<th>Taking base:</th>
<th>$[(1,1), (8, -5)]$</th>
<th>$[(1,1), (5, -2)]$</th>
<th>$[(8, -5), (5, -2)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of base:</td>
<td>$\sqrt{85}$</td>
<td>$5$</td>
<td>$\sqrt{18}$</td>
</tr>
<tr>
<td>Equation base-line:</td>
<td>$6x + 7y = 13$</td>
<td>$3x + 4y = 7$</td>
<td>$x + y = 3$</td>
</tr>
<tr>
<td>Distance from other corner:</td>
<td>$\frac{3}{\sqrt{85}}$</td>
<td>$\frac{3}{5}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\therefore$ Area =</td>
<td>$3$</td>
<td>$3$</td>
<td>$3$</td>
</tr>
</tbody>
</table>

**Blunders**
- B1 Error in translation
- B2 Error in formula for area of a triangle.
- B3 Incorrect subst

**Slips**
- S1 Arithmetic errors
Part (b) 20 (5, 5, 10) marks Att (2, 2, 3)

3(b) \( f \) is the transformation \((x, y) \rightarrow (x', y')\), where \( x' = 4x + 2y \) and \( y' = -3x - y \).

\( K \) is the line \( x + y = 0 \).

(i) Show that \( K \) is its own image under \( f \).

(ii) \( p \) (1, −1) and \( q \) (3, −3) are two points.

Find the ratio \([pq] : [f(p)f(q)]\), giving your answer in its simplest form.

(i) Evaluate \( x \) and \( y \) 5 marks Att 2
Find image 5 marks Att 2

3 (b) (i)

\[
\begin{align*}
x' &= 4x + 2y \\
2y' &= -6x - 2y \\
x' + 2y' &= -2x \quad \Rightarrow \quad x = \frac{1}{2}(-x' - 2y').
\end{align*}
\]

But \( y = \frac{1}{2}x' - 2x \Rightarrow y = \frac{1}{2}x' + x' + 2y' \Rightarrow y = \frac{1}{2}(3x' + 4y').

\( K : x + y = 0 \Rightarrow f(K) : \frac{1}{2}(-x' - 2y') + \frac{1}{2}(3x' + 4y') = 0 \)

\( f(K) : -x' - 2y' + 3x' + 4y' = 0 \Rightarrow 2x' + 2y' = 0 \Rightarrow x' + y' = 0. \)

\( f(K) : x + y = 0 \quad \Rightarrow \quad f(K) = K. \)

Blunders
B1 Error in setting up or solving simultaneous equations.
B2 Error in finding image.

Slips
S1 Arithmetic errors

(b) (ii) 10 marks Att 3

3 (b) (ii)

\[
\begin{align*}
x' &= 4x + 2y \quad \text{and} \quad y' = -3x - y. \\
|pq| &= \sqrt{(1-3)^2 + (-1+3)^2} = \sqrt{8} = 2\sqrt{2}. \\
f(p) &= f(1, -1) = (2, -2) \quad \text{and} \quad f(q) = f(3, -3) = (6, -6). \\
\Rightarrow \ |f(p)f(q)| &= \sqrt{(2-6)^2 + (-2+6)^2} = \sqrt{32} = 4\sqrt{2}. \\
\therefore \ |pq| : |f(p)f(q)| &= 2\sqrt{2} : 4\sqrt{2} = 1 : 2.
\end{align*}
\]

Blunders
B1 Error in distance formula
B2 Error in finding images
B3 No ratio shown or incorrect order.

Slips
S1 Arithmetic errors
Part (c) 20 (10, 5, 5) marks Att (3, 2, 2)

3 (c) Consider the equation \( k(3x - 5y + 6) + l(5x - 7y + 4) = 0 \).

(i) Show that, for any \( k, l \in \mathbb{R} \), the given equation represents a line passing through the point of intersection of \( 3x - 5y + 6 = 0 \) and \( 5x - 7y + 4 = 0 \).

(ii) Find the relationship between \( k \) and \( l \) for which the given equation represents a line of slope 2.

(iii) If \( k = 1 \), what line through the point of intersection cannot be represented by the given equation? Justify your answer.

<table>
<thead>
<tr>
<th>Part (c) (i)</th>
<th>10 marks</th>
<th>Att 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3(c)(i)</td>
<td>The given equation is of first degree in ( x ) and ( y ) and is therefore a line. It remains to show that it passes through the point of intersection. Let ((x_1, y_1)) be the point of intersection of ( 3x - 5y + 6 = 0 ) and ( 5x - 7y + 4 = 0 ). ((x_1, y_1)) is on the line ( 3x - 5y + 6 = 0 ) ( \Rightarrow ) ( 3x_1 - 5y_1 + 6 = 0 ). ((x_1, y_1)) is on the line ( 5x - 7y + 4 = 0 ) ( \Rightarrow ) ( 5x_1 - 7y_1 + 4 = 0 ).</td>
<td></td>
</tr>
</tbody>
</table>

Blunders
B1 Fails to show expression represents a line.
B2 Fails to show passes through point of intersection.

Slips
S1 Arithmetic

<table>
<thead>
<tr>
<th>Part (c) (ii)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>
| 3 (c) (ii)    | \( k(3x - 5y + 6) + l(5x - 7y + 4) = 0 \) \( \Rightarrow \) \( x(3k + 5l) + y(-5k - 7l) + (6k + 4l) = 0 \)
\[ \therefore \text{Slope}=\frac{3k + 5l}{5k + 7l} = 2 \Rightarrow 10k + 14l = 3k + 5l \Rightarrow 7k + 9l = 0. \] |

Blunders
B1 Error in finding slope
B2 Transposing error

Slips
S1 Arithmetic errors.

<table>
<thead>
<tr>
<th>Part (c) (iii)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>
| 3 (c) (iii)    | If \( k = 1 \), the equation \( k(3x - 5y + 6) + l(5x - 7y + 4) = 0 \) cannot represent the line \( 5x - 7y + 4 = 0 \).
Justification: If \( k = 1 \), the slope of the line will be \( \frac{3 + 5l}{5 + 7l} \).
There is no value of \( l \) that can make this expression equal to \( \frac{5}{7} \), (because attempting to solve this yields \( 21 + 35l = 25 + 35l \), which has no solution). |

Blunders
B1 Fails to justify answer.
## QUESTION 4

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5) marks</td>
<td>Att (2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5) marks</td>
<td>Att (2, 2, 2)</td>
</tr>
</tbody>
</table>

### Part (a) 10 (5, 5) marks Att (2, 2)

#### 4. (a) Show that \((\cos A + \sin A)^2 = 1 + \sin 2A\).

**Square** 5 marks Att 2

**Tidy up** 5 marks Att 2

\[
(\cos A + \sin A)^2 = \cos^2 A + \sin^2 A + 2\cos A\sin A = 1 + \sin 2A.
\]

**Blunders**

B1 \(\cos^2 A + \sin^2 A \neq 1\)

B2 \(2\cos A\sin A \neq \sin 2A\)

### Part (b) 20 (5, 5, 5) marks Att (2, 2, 2)

#### 4 (b) Find all the solutions of the equation

\[6 \cos^2 x + \sin x - 5 = 0\], where \(0^\circ \leq x \leq 360^\circ\).

Give the solutions correct to the nearest degree.

**\(\cos^2 x = 1 - \sin^2 x\)** 5 marks Att 2

**Quadratic form** 5 marks Att 2

**Solve quadratic** 5 marks Att 2

**Values for \(x\)** 5 marks Att 2

\[
6\cos^2 x + \sin x - 5 = 6(1 - \sin^2 x) + \sin x - 5 = 0 \Rightarrow 6\sin^2 x - \sin x - 1 = 0.
\]

\[
\therefore (2\sin x - 1)(3\sin x + 1) = 0 \Rightarrow \sin x = \frac{1}{2} \text{ or } \sin x = -\frac{1}{3}.
\]

\[
\therefore x = 30^\circ, 150^\circ, 199^\circ, 341^\circ.
\]

**Blunders**

B1 Incorrect substitution for \(\cos^2 x\)

B2 Error in factors or quadratic formula.

B3 Each incorrect or missing solution.

**Slips**

S1 Arithmetic/not rounded.

**Attempts**

A1 \(\cos^2 x = 1 - \sin^2 x\)
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

4 (c) \([ab]\) is the diameter of a semicircle of centre \(o\) and radius-length \(r\).
\([ac]\) is a chord such that \(\angle cab = \alpha\), where \(\alpha\) is in radian measure.

(i) Find \(|ac|\) in terms of \(r\) and \(\alpha\).

(ii) \([ac]\) bisects the area of the semicircular region.

Show that \(2\alpha + \sin 2\alpha = \frac{\pi}{2}\).

\(\) 5 marks Att 2

4 (c) (i)

\([ab]\) is a diameter, \(\therefore \angle acb = 90^\circ\).
\[\cos \angle cab = \cos \alpha = \frac{|ac|}{|ab|} = \frac{|ac|}{2r} \Rightarrow |ac| = 2r \cos \alpha.\]

\(\) 5 marks Att 2

(ii) Area of triangle 5 marks Att 2

Area of sector 5 marks Att 2

Show 5 marks Att 2

\(\)

Area of semicircle = \(\frac{1}{2} \pi r^2\).

Area of region \(abc\) = Area of triangle \(aoc\) + sector \(obc\).

Area of triangle \(aoc\) = \(\frac{1}{2} |ao||ac| \sin \alpha = \frac{1}{2} r(2r \cos \alpha) \sin \alpha = \frac{1}{2} r^2 \sin 2\alpha\).

Area of sector \(obc\) = \(\frac{1}{2} r^2(2\alpha) = r^2 \alpha\). as \(|\angle cob| = 2\alpha\), since \(|ao| = |oc| \Rightarrow \angle aco = \alpha\).

\(\therefore\) Area of region = \(r^2 \alpha + \frac{1}{2} r^2 \sin 2\alpha\).

This is half the semicircle, so \(r^2 \alpha + \frac{1}{2} r^2 \sin 2\alpha = \frac{1}{4} \pi r^2 \Rightarrow 2\alpha + \sin 2\alpha = \frac{\pi}{2}\).

\(\)

Blunders
B1 Error in area of triangle
B2 Error in area of sector

Slips
S1 Arithmetic
QUESTION 5

Part (a) 10 marks Att 3

Part (b) 20 (10, 10) marks Att (3, 3)

Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

5. (a) Evaluate \( \lim_{x \to 0} \frac{\sin 2x}{\sin 3x} \).

\[
5\text{ (a) } \lim_{x \to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \to 0} \left( \frac{\sin 2x}{2x} \cdot \frac{2x}{\sin 3x} \cdot \frac{3x}{3} \right) = \lim_{x \to 0} \left( \frac{\sin 2x}{2x} \cdot \lim_{x \to 0} \left( \frac{2x}{\sin 3x} \cdot \frac{3x}{3} \right) \right) = \frac{2}{3} \cdot \lim_{x \to 0} \frac{\sin 2x}{2x} = \frac{2}{3} \cdot 1 = \frac{2}{3}.
\]

* Accept correct answer with or without work; if answer is correct, ignore the work.

Blunders
B1 \( \sin 2x = 2\sin x \).
B2 Error in differentiation

Slips
S1 Arithmetic

Part (b) 20 (10, 10) marks Att (3, 3)

5 (b) Using the formula \( \cos(A + B) = \cos A \cos B - \sin A \sin B \), derive a formula for \( \cos(A - B) \) and hence prove that \( \sin(A + B) = \sin A \cos B + \cos A \sin B \).

Formula for \( \cos(A - B) \)

10 marks Att 3

Hence prove

10 marks Att 3

5 (b) \( \cos(A + B) = \cos A \cos B - \sin A \sin B \Rightarrow \cos(A - B) = \cos A \cos(-B) - \sin A \sin(-B) \).

\[ \therefore \cos(A - B) = \cos A \cos B + \sin A \sin B \ as \ \cos(-B) = \cos B \ and \ \sin(-B) = -\sin B. \]

\[ \sin(A + B) = \cos(90^\circ - [A + B]) = \cos \left( 90^\circ - A - B \right) = \cos(90^\circ - A) \cos B + \sin(90^\circ - A) \sin B = \sin A \cos B + \cos A \sin B \]

as \( \cos(90^\circ - A) = \sin A \) and \( \sin(90^\circ - A) = \cos A \).

Blunders
B1 Fails to replace B with -B
B2 Fails to show \( \cos(-B) = \cos B \)
B3 Fails to show \( \sin(-B) = -\sin B \)
B4 Hence not used

Slips
S1 Arithmetic error
Part (c) 20 (5, 5, 5, 5) marks  
Att (2, 2, 2, 2)

5 (c)  
$p, q$ and $r$ are three points on horizontal ground.  
$[sr]$ is a vertical pole of height $h$ metres.  
The angle of elevation of $s$ from $p$ is $60^\circ$ and the angle of elevation of $s$ from $q$ is $30^\circ$.  
$|pq| = c$ metres.  
Given that $3c^2 = 13h^2$, find $\angle prq$.

![Diagram with points p, q, r, and s, where s is a vertical pole, p and q are points on horizontal ground, and the angles of elevation from p and q to s are 60° and 30° respectively.]

Calculates $|pr|$ 5 marks Att 2  
Calculates $|rq|$ 5 marks Att 2  
Cosine rule 5 marks Att 2  
Solves equation 5 marks Att 2

\[
\begin{align*}
\tan 60^\circ &= \frac{h}{|pr|} \quad \Rightarrow \quad |pr| = \frac{h}{\tan 60^\circ} = \frac{h}{\sqrt{3}}. \\
\tan 30^\circ &= \frac{h}{|rq|} \quad \Rightarrow \quad |rq| = \frac{h}{\tan 30^\circ} = \frac{h}{\frac{1}{\sqrt{3}}} = h\sqrt{3}. \\
cos \angle prq &= \frac{|pr|^2 + |rq|^2 - |pq|^2}{2|pr||rq|} = \frac{\frac{h^2}{3} + 3h^2 - c^2}{2h^2} \\
&= \frac{10h^2 - 3c^2}{6h^2} = \frac{10h^2 - 13h^2}{6h^2} = -\frac{1}{2}. \quad \therefore \angle prq = 120^\circ.
\end{align*}
\]

Blunders  
B1 Error in trig ratio  
B2 Error in Cosine Rule  
B3 Error in solving equation

Slips  
S1 Arithmetic errors.
**QUESTION 6**

| Part (a) | 10 (5, 5) marks | Att (-, 2) |
| Part (b) | 20 (5, 5, 5, 5) marks | Att (2, 2, 2, 2) |
| Part (c) | 20 (10, 5, 5) marks | Att (3, 2, 2) |

6. (a) Six people, including Mary and John, sit in a row.
   (i) How many different arrangements of the six people are possible?
   (ii) In how many of these arrangements are Mary and John next to each other?

(a) (i) 5 marks Hit/Miss

| 6 (a) (i) | Number of arrangements $=^6P_6 = 720$. |

(a) (ii) 5 marks Att 2

| 6 (a) (ii) | Number of arrangements $=^5P_3 \times ^2P_2 = 240$. |

**Blunders**
B1 Fails to rearrange Mary and John
B2 Uses $5! + 2!$

**Slips**
S1 Arithmetic

**Part (b) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)**

6 (b) $\alpha$ and $\beta$ are the roots of the quadratic equation $px^2 + qx + r = 0$.
   $u_n = l\alpha^n + m\beta^n$, for all $n \in \mathbb{N}$.
   Show that $pu_{n+2} + qu_{n+1} + ru_n = 0$, for all $n \in \mathbb{N}$.

Uses root property correctly 5 marks Att 2
Deduces $u_{n+1}, u_{n+2}$ 5 marks Att 2
Substitutes and tidies up 5 marks Att 2
Conclusion 5 marks Att 2

| 6 (b) | $\alpha$ is a root of $px^2 + qx + r = 0$ \(\Rightarrow p\alpha^2 + q\alpha + r = 0\)  |
|       | Similarly: $p\beta^2 + q\beta + r = 0$  |
|       | Given: $u_n = l\alpha^n + m\beta^n \Rightarrow u_{n+1} = l\alpha^{n+1} + m\beta^{n+1}, u_{n+2} = l\alpha^{n+2} + m\beta^{n+2}$  |
|       | $\Rightarrow pu_{n+2} + qu_{n+1} + ru_n$  |
|       | $= p[l\alpha^{n+2} + m\beta^{n+2}] + q[l\alpha^{n+1} + m\beta^{n+2}] + r[l\alpha^n + m\beta^n]$  |
|       | $= l\alpha^n[p\alpha^2 + q\alpha + r] + m\beta^n[p\beta^2 + q\beta + r]$  |
|       | $= l\alpha^n[0] + m\beta^n[0]$  |
|       | $= 0$  |
**Blunders**

B1 Fails to use root property  
B2 Error in expressing value of term  
B3 Error in substituting or tidying  
B4 No conclusion

---

**Part (c)**

<table>
<thead>
<tr>
<th>20 (10, 5, 5) marks</th>
<th>Att (3, 2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>6 (c)</strong> w white discs and r red discs are placed in a box. Two of the discs are drawn at random from the box. The probability that both discs are red is ( p ).</td>
<td></td>
</tr>
<tr>
<td>(i) Find ( p ) in terms of ( w ) and ( r ).</td>
<td></td>
</tr>
<tr>
<td>(ii) When ( w = 1 ), find the value of ( r ) for which ( p = \frac{1}{2} ).</td>
<td></td>
</tr>
<tr>
<td>(iii) There are other values of ( w ) and ( r ) that also give ( p = \frac{1}{2} ).</td>
<td></td>
</tr>
</tbody>
</table>

The next smallest such value of \( w \) is even.  
By investigating the even numbers in turn, find this value of \( w \) and the corresponding value of \( r \).

---

**Blunders**

B1 Incorrect possible  
B2 Incorrect favourable  
B3 No division

---

**Slips**

S1 Arithmetic

---

**6 (c) (i)** There are \((r + w)\) discs in the box.  
\[
\therefore \text{Number of ways of picking two discs} = \binom{r+w}{2} = \frac{(r + w)(r + w - 1)}{2}. 
\]

Number of ways of picking two red discs = \( \binom{r}{2} = \frac{r(r-1)}{2} \).  
\[
\therefore \text{Probability} = \frac{r(r-1)}{(r + w)(r + w - 1)}. 
\]

---

**Blunders**

B1 Error in solving the equation

---

**Slips**

S1 Arithmetic

---

**6 (c) (ii)**  
\[
\frac{r(r-1)}{(r+1)r} = \frac{1}{2} \Rightarrow 2(r-1) = r + 1 \Rightarrow r = 3. 
\]
(c) (iii) \[ \text{Probability} = \frac{r(r-1)}{(r+w)(r+w-1)} = \frac{1}{2}, \text{ for } w = 2, 4, 6, \text{ etc.} \]

\[ w = 2 \Rightarrow \frac{r(r-1)}{(r+2)(r+1)} = \frac{1}{2} \Rightarrow 2r^2 - 2r = r^2 + 3r + 2 \Rightarrow r^2 - 5r - 2 = 0. \]

No solution possible for \( r \) a natural number.

\[ w = 4 \Rightarrow \frac{r(r-1)}{(r+4)(r+3)} = \frac{1}{2} \Rightarrow 2r^2 - 2r = r^2 + 7r + 12 \Rightarrow r^2 - 9r - 12 = 0. \]

No solution possible for \( r \) a natural number.

\[ w = 6 \Rightarrow \frac{r(r-1)}{(r+6)(r+5)} = \frac{1}{2} \Rightarrow 2r^2 - 2r = r^2 + 11r + 30 \Rightarrow r^2 - 13r - 30 = 0. \]

\[ \therefore (r-15)(r+2) = 0 \Rightarrow r = 15 \text{ as } r \neq -2. \] \[ \therefore r = 15 \text{ and } w = 6. \]

**Blunders**

B1 Error in setting up or solving the equation

**Slips**

S1 Arithmetic
QUESTION 7

Part (a)  
10 (5, 5) marks  
Att (-, 2)

Part (b)  
20 (10, 10) marks  
Att (3, 3)

Part (c)  
20 (5, 5, 5, 5) marks  
Att (2, 2, 2, 2)

7.

(a) (i)  
How many different selections of four letters can be made from the letters of the word FLORIDA?

(ii) How many of these selections contain at least one vowel?

(a) (i)  
5 marks  
Hit/Miss

7 (a) (i)  
$^7C_4 = 35$.

(a) (ii)  
5 marks  
Att 2

7 (a) (ii)  
Number of selections of four letters with no vowel $= ^4C_4 = 1$.

Number of selections with at least one vowel $= 35 - 1 = 34$.

Blunders
B1 Error in deriving solution with no vowel
B2 Does not subtract

Part (b)  
20 (10, 10) marks  
Att (3, 3)

7 (b) Two dice are thrown.

(i) What is the probability of getting two identical numbers or a total of five?

(ii) What is the probability that the product of the two numbers thrown is at least twice their sum?

(b) (i)  
10 marks  
Att 3

7 (b) (i)  
Number of possible outcomes $= 6 \times 6 = 36$.

Outcomes of interest: (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (2, 3), (3, 2).

So there are 10 outcomes of interest.

Probability $= \frac{10}{36} = \frac{5}{18}$.

Blunders
B1 Incorrect possible
B2 Incorrect favourable
B3 No division

Slips
S1 Arithmetic
### (b) (ii) 10 marks Att 3

<table>
<thead>
<tr>
<th>Outcomes of interest</th>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6, 6)</td>
<td>36</td>
<td>12</td>
</tr>
<tr>
<td>(6, 5) or (5, 6)</td>
<td>30</td>
<td>11</td>
</tr>
<tr>
<td>(6, 4) or (4, 6)</td>
<td>24</td>
<td>10</td>
</tr>
<tr>
<td>(6, 3) or (3, 6)</td>
<td>18</td>
<td>9</td>
</tr>
<tr>
<td>(5, 5)</td>
<td>25</td>
<td>10</td>
</tr>
<tr>
<td>(5, 4) or (4, 5)</td>
<td>20</td>
<td>9</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>16</td>
<td>8</td>
</tr>
</tbody>
</table>

11 outcomes of interest: \[\therefore \text{Probability} = \frac{11}{36}.\]

**Blunders**

B1 Incorrect possible
B2 Incorrect favourable
B3 No division

**Slips**

S1 Arithmetic

### Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

#### (c)(i) Correct Total Mean 5 marks Att 2

<table>
<thead>
<tr>
<th>Mean</th>
<th>5 marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{7}{2} (2a + 6d) = 7a + 21d )</td>
<td>( \Rightarrow \text{mean} = \bar{x} = \frac{7a + 21d}{7} = a + 3d. )</td>
</tr>
</tbody>
</table>

**Blunders**

B1 Error in finding total
B2 Error in finding mean

**Slips**

S1 Arithmetic

#### (c)(ii) Correct total devs Correct std dev 5 marks Att 2

\[ \bar{x} = a + 3d \Rightarrow -3d, -2d, -d, 0, d, 2d, 3d \]

\[ \sum d^2 = 28d^2. \quad \therefore \text{Standard deviation} = \sqrt{\frac{28d^2}{7}} = 2d. \]

**Blunders**

B1 Error in finding total
B2 Error in finding Std Dev

**Slips**

S1 Arithmetic
### QUESTION 8

<table>
<thead>
<tr>
<th>Part (a)</th>
<th>10 (5, 5) marks</th>
<th>Att (2, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part (b)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
<tr>
<td>Part (c)</td>
<td>20 (5, 5, 5, 5) marks</td>
<td>Att (2, 2, 2, 2)</td>
</tr>
</tbody>
</table>

**8. (a)**  
$p$ and $q$ are real numbers such that $p + q = 1$. Find the value of $p$ that maximizes the product $pq$.

**Expression in one variable**  
5 marks  
Att 2

**Finishes**  
5 marks  
Att 2

**8 (a)**  
$pq = p(1 - p) = p - p^2$

$$\frac{dp}{dp}(p - p^2) = 1 - 2p = 0 \text{ for maximum } \Rightarrow p = \frac{1}{2}.$$  
$$\frac{d^2}{dp^2}(p - p^2) = -2 < 0 \Rightarrow \text{ maximum value at } p = \frac{1}{2}.$$  

**Blunders**  
B1 Error in finding expression  
B2 Error in finding derivative

**Slips**  
S1 Arithmetic

<table>
<thead>
<tr>
<th>Part (b)</th>
<th>20 (5, 5, 5, 5) marks</th>
<th>Att (2, 2, 2, 2)</th>
</tr>
</thead>
</table>

**8 (b) (i)** Derive the Maclaurin series for $f(x) = (1 + x)^m$ up to and including the term containing $x^3$.

**8 (b) (ii)** Given that the general term of the series $f(x)$ is

$$\frac{m(m-1)(m-2)........(m-r+1)}{r!} x^r,$$

show that the series converges for $-1 < x < 1$.

**8 (b) (i)**  
$f(x) = (1 + x)^m \Rightarrow f(0) = 1.$

$f'(x) = m(1 + x)^{m-1} \Rightarrow f'(0) = m.$

$f''(x) = m(m - 1)(1 + x)^{m-2} \Rightarrow f''(0) = m(m - 1).$

$f'''(x) = m(m - 1)(m - 2)(m - 3) x^{m-3} \Rightarrow f'''(0) = m(m - 1)(m - 2).$

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ..........$$

$$\therefore f(x) = (1 + x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + .......$$
**Blunders**

B1 Error in finding expression
B2 Error in finding Derivative
B3 Error in establishing series

**Slips**

S1 Arithmetic

<table>
<thead>
<tr>
<th>Part (b) (ii)</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
</table>

8 (b) (ii)

\[
 u_{r+1} = \frac{m(m-1)(m-2)\ldots\ldots(m-r+1)}{r!} x^r \quad \Rightarrow \quad u_r = \frac{m(m-1)(m-2)\ldots\ldots(m-r)}{(r-1)!} x^{r-1}
\]

\[
 \lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} \frac{m(m-1)(m-2)\ldots\ldots(m-r+1)}{r!} x^r \times \frac{(r-1)!}{m(m-1)(m-2)\ldots\ldots(m-r)} \frac{1}{x^{r-1}}
\]

\[
= \lim_{r \to \infty} \frac{(m-r+1)x^r}{r} = \lim_{r \to \infty} \left| \frac{mx}{r} - x + \frac{x}{r} \right| = |x| < 1 \quad \Rightarrow \quad |x| < 1 \quad \text{for convergency.}
\]

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2</td>
<td>1 1</td>
<td>1</td>
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<tr>
<td>1 2</td>
<td>1 1</td>
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<td>1 2</td>
<td>1 1</td>
<td>1</td>
</tr>
<tr>
<td>1 2</td>
<td>1 1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
 \Rightarrow -1 < x < 1.
\]

**Blunders**

B1 Error in finding expression
B2 Error in finding Derivative
B3 Error in establishing range.

**Slips**

S1 Arithmetic
8 (c) Evaluate \[ \int_{0}^{1} \tan^{-1} x \, dx. \]

Set up integration by parts \hspace{1cm} 5 marks \hspace{1cm} Att 2
Parts stage done \hspace{1cm} 5 marks \hspace{1cm} Att 2
\[ \int \frac{x}{1+x^2} \, dx \] \hspace{1cm} 5 marks \hspace{1cm} Att 2
Evaluation \hspace{1cm} 5 marks \hspace{1cm} Att 2

\[
\begin{align*}
\int u \, dv &= uv - \int v \, du, \quad \text{Let } u = \tan^{-1} x \quad \text{and} \quad dv = dx. \quad \therefore \quad du = \frac{1}{1 + x^2} \, dx \quad \text{and} \quad v = x. \\
\therefore \int_{0}^{1} \tan^{-1} x \, dx &= x \tan^{-1} x \Big|_{0}^{1} - \int_{0}^{1} \frac{x}{1 + x^2} \, dx. \\
\int_{0}^{1} \frac{x}{1 + x^2} \, dx &= \frac{1}{2} \int_{1}^{2} \frac{dw}{w} = \frac{1}{2} \left[ \log_e x \right]_{1}^{2} = \frac{1}{2} \left[ \log_e 2 - \log_e 1 \right] = \frac{1}{2} \log_e 2.
\end{align*}
\]

\[
\therefore \int_{0}^{1} \tan^{-1} x \, dx = \left[ x \tan^{-1} x \right]_{0}^{1} - \frac{1}{2} \log_e 2 = \tan^{-1} 1 - \frac{1}{2} \log_e 2 = \frac{\pi}{4} - \frac{1}{2} \log_e 2.
\]

**Blunders**
B1 Error in setting up expression
B2 Error in integrating
B3 Error in establishing value

**Slips**
S1 Arithmetic
Note: Candidates may attempt to use a Maclaurin expansion to answer this question. They are unlikely to make substantial progress. The solution below is presented so as to facilitate the award of relevant partial credit.

Expand \( \tan^{-1} x \) \\
Integrate & evaluate at limits \\
Partial fractions & separate 2 series \\
Finish

<table>
<thead>
<tr>
<th>Expand ( \tan^{-1} x )</th>
<th>5 marks</th>
<th>Att 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrate &amp; evaluate at limits</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
<tr>
<td>Partial fractions &amp; separate 2 series</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
<tr>
<td>Finish</td>
<td>5 marks</td>
<td>Att 2</td>
</tr>
</tbody>
</table>

\[
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots
\]

\[
\therefore \int_0^1 \tan^{-1} x \, dx = \frac{x^2}{1.2} - \frac{x^4}{3.4} + \frac{x^6}{5.6} - \frac{x^8}{7.8} + \cdots + (-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} + \cdots
\]

\[
= \frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \cdots + (-1)^n \frac{1}{(2n+1)(2n+2)} + \cdots
\]

\[
= \left( \frac{1}{1} - \frac{1}{2} \right) - \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) - \left( \frac{1}{7} - \frac{1}{8} \right) + \cdots + (-1)^n \left( \frac{1}{2n+1} - \frac{1}{2n+2} \right) + \cdots
\]

\[
= \tan^{-1} \frac{1}{2} \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1} + \cdots \right)
\]

\[
= \tan^{-1} \frac{1}{2} \ln(1 + 1)
\]

\[
= \frac{\pi}{4} - \frac{1}{2} \ln 2.
\]
**QUESTION 9**

| Part (a) | 10 (5, 5) marks | Att (2, 2) |
| Part (b) | 20 (10, 10) marks | Att (3, 3) |
| Part (c) | 20 (5, 5, 5, 5) marks | Att (2, 2, 2, 2) |

9. (a) Two events $E_1$ and $E_2$ are independent. $P(E_1) = \frac{1}{5}$ and $P(E_2) = \frac{1}{7}$. Find

(i) $P(E_1 \cap E_2)$

(ii) $P(E_1 \cup E_2)$.

(a) (i) 5 marks Att 2

\[
P(E_1 \cap E_2) = P(E_1)P(E_2) = \frac{1}{5} \times \frac{1}{7} = \frac{1}{35}.
\]

*Blunders*

B1   Addition for multiplication

*Slips*

S1   Arithmetic

(a) (ii) 5 marks Att 2

\[
P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{1}{5} + \frac{1}{7} - \frac{1}{35} = \frac{11}{35}.
\]

*Blunders*

B1   Multiplication for addition

B2   Double counts

*Slips*

S1   Arithmetic
Part (b) | 20 (10, 10) marks | Att (3, 3)
---|---|---

9 (b) Five unbiased coins are tossed.

(i) Find the probability of getting three heads and two tails.

(ii) The five coins are tossed eight times. Find the probability of getting three heads and two tails exactly four times.

Give your answer correct to three places of decimals.

Part (b) (i) | 10 marks | Att 3
---|---|---

9 (b) (i) \[ p = \frac{1}{2}, \quad q = \frac{1}{2} \Rightarrow \text{Probability} = \binom{5}{3} \left( \frac{1}{2} \right)^3 \left( \frac{1}{2} \right)^2 = \frac{10}{32} = \frac{5}{16}. \]

**Blunders**
B1 Error in finding \( p \) or \( q \)
B2 Error in finding Binomial Coefficients
B3 Error in evaluation

**Slips**
S1 Arithmetic

Part (b) (ii) | 10 marks | Att 3
---|---|---

9 (b) (ii) \[ p = \frac{5}{16}, \quad q = \frac{11}{16} \Rightarrow \text{Probability} = \binom{8}{4} \left( \frac{5}{16} \right)^4 \left( \frac{11}{16} \right)^4 = 0.149. \]

**Blunders**
B1 Error in finding \( p \) or \( q \)
B2 Error in binomial coefficients

**Slips**
S1 Arithmetic
9 (c) The amounts due on monthly mobile phone bills in Ireland are normally distributed with mean €53 and standard deviation €15.

(i) If a monthly phone bill is chosen at random, find the probability that the amount due is between €47 and €74.

(ii) A random sample of 900 mobile phone bills is taken. Find the probability that the mean amount due on the bills in the sample is greater than €53.3.

\[ \bar{x} = 53, \sigma = 15. \]
\[ P(47 < x < 74) = P(z_1 < z < z_2) \]
\[ z_1 = \frac{x - \bar{x}}{\sigma} = \frac{47 - 53}{15} = -\frac{6}{15} = -0.4. \]
\[ z_2 = \frac{74 - 53}{15} = \frac{21}{15} = 1.4. \]
\[ P(-0.4 < z < 1.4) = P(z \leq 1.4) - P(z > 0.4) = 0.9192 - [1 - P(z \leq 0.4)] \]
\[ = 0.9192 - (1 - 0.6554) = 0.9192 - 0.3446 = 0.5746. \]

Blunders
B1 Error in finding \( z_1 \) or \( z_2 \)
B2 Error in setting up probability
B3 Error in evaluation

Slips
S1 Arithmetic

(ii) Std error
\[ \sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{900}} = \frac{15}{30} = \frac{1}{2} \]
\[ P(x > 53.3) = P\left( z > \frac{53.3 - 53}{0.5} \right) = P(z > 0.6) = 1 - P(z \leq 0.6) \]
\[ = 1 - 0.7257 = 0.2743. \]

Blunders
B1 Error in finding standard error
B2 Error in evaluation

Slips
S1 Arithmetic
QUESTION 10

Part (a) 10 (5, 5) marks Att (-, -)
Part (b) 20 (10, 10) marks Att (3, 3)
Part (c) 20 (10, 10) marks Att (3, 3)

Part (a) 10 (5, 5) marks Att (-, -)

10. (a) For each of the following, give a reason why it is not a group.
   (i) The set of natural numbers under subtraction.
   (ii) The set of real numbers under multiplication.

(a) (i) 5 marks Hit/Miss

10 (a) (i) Not closed: e.g. \(6 - 14 = -8\), \(-8 \notin \mathbb{N}\).

(a) (ii) 5 marks Hit/Miss

10 (a) (ii) Not all elements have inverses: \(0 \in \mathbb{R}\), but 0 has no multiplicative inverse in \(\mathbb{R}\).

Part (b) 20 (10, 10) marks Att (3, 3)

10 (b) \(G = \{ I_\pi, R_{180^\circ}, S_X, S_Y \}\) is the set of symmetries of the rectangle \(abcd\).

(i) Show that \(G\) is a group under composition.
You may assume that composition is associative.

(ii) Find \(Z(G)\), the centre of the group.

Part (b) (i) 10 marks Att 3

10 (b) (i) 

<table>
<thead>
<tr>
<th></th>
<th>(I_{\pi})</th>
<th>(R_{180^\circ})</th>
<th>(S_X)</th>
<th>(S_Y)</th>
</tr>
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<tr>
<td>(I_{\pi})</td>
<td>(I_{\pi})</td>
<td>(R_{180^\circ})</td>
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<td>(S_Y)</td>
<td>(S_X)</td>
<td>(R_{180^\circ})</td>
<td>(I_{\pi})</td>
</tr>
</tbody>
</table>

Closed: No new element.

Associative: yes, given.

Identity: \(I_{\pi}\)

Inverses: \((I_{\pi})^{-1} = I_{\pi}\), \((R_{180^\circ})^{-1} = R_{180^\circ}\), \((S_X)^{-1} = S_X\), \((S_Y)^{-1} = S_Y\).

Blunders
B1 Identity not given
B2 Inverses not stated
B3 Closure not defined

Slips
S1 each inverse not given
Part (b) (ii)  

**10 (b) (ii)** In table elements are symmetrical about main diagonal.

\[ \therefore G \text{ is a commutative group } \Rightarrow G(Z) = G = \{ I, R_{180}^{-}, S_{X}, S_{Y} \}. \]

or from the table, \( x \circ y = y \circ x \) for all \( x, y \in G \) i.e each element commutes with each other element so \( Z(G) = G \).

Blunders
B1 Each element missing from set.

Part (c)  

**20 (10, 10) marks**

\[ \text{(i)} \quad \text{any group of prime order is cyclic.} \]

\[ \text{(ii)} \quad \text{the order of any element of a finite group } G \text{ divides the order of } G. \]

Part (c) (i)  

**10 marks**

\[ \text{(i)} \quad \text{Let } (G, *) \text{ be a group of order } k, \text{ where } k \text{ is prime.} \]

Let \( a \in G \) and \( a \neq e \). \[ \therefore \langle a \rangle, \text{ (the group generated by } a) \text{ is a subgroup of } G. \]

Hence, the order of \( \langle a \rangle \) is a factor of \( k \) (by Lagrange’s theorem).

But \( k \) is prime \[ \Rightarrow \text{ order of } \langle a \rangle = k, \text{ (since the order of } \langle a \rangle \neq 1, \text{ since } a \neq e). \]

\[ \therefore \langle a \rangle = G. \quad \therefore G \text{ is cyclic.} \]

Blunders
B1 Fails to establish that \( < a > \) is a subgroup of \( G \).
B2 Fails to use Lagrange
B3 No conclusion

Part (c) (ii)  

**10 marks**

\[ \text{(ii)} \quad \text{Let } (G, *) \text{ be a group of order } n. \]

Let \( a \in G \) and let the order of \( a \) be \( m \).

Then \( m \) is also the order of \( <a> \), the subgroup generated by \( a \).

But the order of a subgroup divides the order of the group (by Lagrange’s theorem).

\[ \therefore m \text{ is a factor of } n. \]

That is, the order of the element \( a \) divides the order of the group \( G \).

Blunders
B1 Fails to use Lagrange
B2 No conclusion.
**QUESTION 11**

**Part (a)** 10 marks  
11. (a) Find the eccentricity of an ellipse with equation \( \frac{x^2}{64} + \frac{y^2}{48} = 1 \).

\[
a^2 = 64, \ b^2 = 48 \quad \text{and} \quad b^2 = a^2 \left(1 - e^2\right).
\]

\[
48 = 64 \left(1 - e^2\right) \Rightarrow 64e^2 = 16 \Rightarrow e^2 = \frac{1}{4} \quad \therefore e = \frac{1}{2}.
\]

**Blunders**

B1 Incorrect \( a^2 \)
B2 \( b^2 \neq a^2 \left(1 - e^2\right) \)

**Slips**

S1 Arithmetic

**Part (b)** 20 (5, 5, 5, 5) marks  
11 (b) Prove that a similarity transformation maps the orthocentre of a triangle onto the orthocentre of the image triangle.

**Orthocentre** 5 marks  
**Mapping** 5 marks  
**Perp invariant** 5 marks  
**Conclusion** 5 marks

11 (b)

\[
f \quad \text{is a similarity transformation.}
\]

\[
[pq] \perp [qr] \quad \text{and} \quad [rt] \perp [pq] \quad \Rightarrow \quad h \quad \text{is the orthocentre of triangle } pqr.
\]

By \( f \), triangle \( pqr \) is mapped to triangle \( p'q'r' \).

To Prove: \( f(h) \) is orthocentre of triangle \( p'q'r' \).

By \( f \), \( [ps] \) maps to \( [p's'] \) and \( [rt] \) maps to \( [r't'] \).

But \( [ps] \perp [qr] \Rightarrow [p's'] \perp [q'r'] \) as perpendicularity is invariant.

Similarly \( [r't'] \perp [p'q'] \) \( \therefore h' \) is orthocentre of triangle \( p'q'r' \).

But \( h = [ps] \cap [rt] \) maps to \( f(h) = [p's'] \cap [r't'] \)

\( \therefore f(h) = h' \Rightarrow f(h) \) is orthocentre of triangle \( p'q'r' \).
Blunders
B1 Fails to define orthocentre
B2 Fails to state perpendicularity is invariant
B3 Fails to show h maps onto f(h)

Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

11 (c) $E$ is the ellipse $\frac{x^2}{4} + y^2 = 1$ and $L$ is the line $y = x$.

Using a transformation that maps $E$ to the unit circle, or otherwise, find the equation of the diameter that is conjugate to $L$ in $E$.

Blunders
B1 Incorrect transformation
B2 Incorrect image for $x - y = 0$
B3 Error in mapping back to the ellipse.

Slips
S1 Arithmetic
Marcanna Breise as ucht freagairt trí Ghaeilge

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronadh ar iarrthóirí nach ngnóthaíonn thar 75% d’iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú síos.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéar ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an gnáthráta 5% i gcás marcanna suas go 225. (e.g. 198 marks \( \times 5\% = 9\cdot9 \Rightarrow \text{bónas} = 9\text{ marc.} \)

Thar 225, is féidir an bónas a riomh de réir na foirmle seo: \([300 – \text{bunmharc}] \times 15\%\), (agus an marc sin a shlánú síos). In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thios.

<table>
<thead>
<tr>
<th>Bunmharc</th>
<th>Marc Bónais</th>
</tr>
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<tbody>
<tr>
<td>226</td>
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<td>227 – 233</td>
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