Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate 2011

Marking Scheme

MATHEMATICS

Higher Level

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GENERAL GUIDELINES FOR EXAMINERS – PAPER 1

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2, …etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5·50 may be written as €5,50.

QUESTION 1

Part (a)	**15 (10, 5) marks**	**Att (3, 2)**
Part (b)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**15 (10, 5) marks**	**Att (3, 2)**

1. **(a)** Simplify fully $\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4}{x^2-1}$

Setting up fraction	**10 marks**	**Att 3**
Fully simplified	**5 marks**	**Att 2**

1 (a)

$$\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4}{x^2-1}=\frac{(x+1)(x+1)-(x-1)(x-1)-4}{(x+1)(x-1)}=\frac{x^2+2x+1-x^2+2x-1-4}{(x+1)(x-1)}$$

$$=\frac{4x-4}{(x+1)(x-1)}=\frac{4(x-1)}{(x+1)(x-1)}=\frac{4}{x+1}$$

Blunders (-3)

B1 Factors once only
B2 Indices
B3 Incorrect cancellation

Part (b)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

1 (b)

(i) Prove the factor theorem for polynomials of degree 2.
That is, given that $f(x)=ax^2+bx+c$ and k is a number such that $f(k)=0$, prove that $(x-k)$ is a factor of $f(x)$.

(ii) The factor theorem also holds for polynomials of higher degree.
Find the values of n for which $(x+k)$ is a factor of the polynomial $g(x)=x^n+k^n$, where $k\neq 0$.

(b) (i) $f(x)-f(k)$ factorised	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1 (b) (i)

$$f(x)=ax^2+bx+c.$$
$$f(k)=ak^2+bk+c.$$
$$\therefore\ f(x)-f(k)=a(x^2-k^2)+b(x-k)=a(x+k)(x-k)+b(x-k).$$
$$\therefore\ f(x)-f(k)=(x-k)(ax+ak+b).$$
$\therefore\ (x-k)$ is a factor of $f(x)-f(k)$.

But $f(k)=0,\ \Rightarrow\ (x-k)$ is a factor of $f(x)$.

Blunders (-3)

B1 Indices

B2 Factors

B3 $f(k) \neq 0$

Slips (-1)

S1 Numerical

OR

(b) (i) Setting up division **5 marks** **Att 2**

Finish **5 marks** **Att 2**

1 (b) (i)

$$f(x) = ax^2 + bx + c$$

$$f(k) = ak^2 + bk + c$$

$$f(x) - f(k) = ax^2 + b - ak^2 - bk$$

$$\begin{array}{r|l} & ax + (ak+b) \\ \hline x-k & ax^2 + bx - ak^2 - bk \\ & \underline{ax^2 - akx} \\ & (ak+b)x - ak^2 - bk \\ & \underline{(ak+b)x - ak^2 - bk} \\ & \qquad\qquad 0 \end{array}$$

But $f(k) = 0,$

$\Rightarrow f(x) = (x-k)[ax + (ak+b)]$

Blunders (-3)

B1 Indices

Slips (-1)

S1 Numerical

S2 Not changing sign when subtracting in division

(b) (ii) **5 marks** **Att 2**

1 (b) (ii)

$(x+k)$ is a factor of $g(x) \;\Rightarrow\; g(-k) = 0.$

$\therefore\; (-k)^n + k^n = 0 \;\Rightarrow\; (-1)^n k^n + k^n = 0.$

$\therefore\; n$ is odd $\;\Rightarrow\; n = \{1, 3, 5, 7, 9, ...\}.$

Blunders (-3)

B1 Deduction root from factor

B2 Indices

B3 $(-1)^n$

B4 Solution set not stated

B5 Only one value n

Part (c) 20(5, 5, 5, 5) marks Att (2, 2, 2, 2)

1 (c) $(x-a)^2$ is a factor of $2x^3-5ax^2+8abx-36a$, where $a \neq 0$.
Find the possible values of a and b.

Set up division	**5 marks**	**Att 2**
Remainder = 0	**5 marks**	**Att 2**
Co-efficients = 0	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1 (c)

$$(x-a)^2 = x^2-2ax+a^2.$$

$$\begin{array}{r|l}
 & 2x-a \\ \hline
x^2-2ax+a^2 & 2x^3-5ax^2+8abx-36a \\
 & \underline{2x^3-4ax^2+2a^2x} \\
 & \quad -ax^2-2a^2x+8abx-36a \\
 & \quad \underline{-ax^2+2a^2x-a^3} \\
 & \qquad -4a^2x+8abx-36a+a^3
\end{array}$$

$$\therefore \left(-4a^2+8ab\right)x+\left(a^3-36a\right)=0.$$

$$\therefore -4a^2+8ab=0 \Rightarrow a-2b=0 \text{ and } a^2-36=0, \text{as } a \neq 0.$$

$$\therefore a=\pm 6 \text{ and } b=\pm 3.$$

$$\text{ie } a=6 \text{ and } b=3 \text{ or } a=-6 \text{ and } b=-3.$$

Blunders (-3)

B1 Expansion of $(x-a)^2$ once only

B2 Indices

B3 Not like to like when equating coefficients

B4 Not two values of 1st variable

Slips (-1)

S1 Not changing sign when subtracting

Attempts

A1 Any effort at division for 2 marks only

A2 (x-a) as factor.

OR

Other factor	**5 marks**	**Att 2**
Correct multiplication	**5 marks**	**Att 2**
Equating coefficients	**5 marks**	**Att 2**
Values	**5 marks**	**Att 2**

1 (c)

One factor $= (x^2 - 2ax + a^2)$

Other factor $= (2x - {}^{36}\!/_{a})$

$$(x^2 - 2ax + a^2).(2x - {}^{36}\!/_{a}) = 2x^3 - 5ax^2 + 8abx - 36a$$

$$2x^3 - 4ax^2 + 2a^2x - \frac{36}{a}x^2 + 72x - 36a = 2x^3 + (-5a)x^2 + 8abx - 36a$$

$$2x^3 + (-4a - {}^{36}\!/_{a})x^2 + (2a^2 + 72)x - 36a = 2x^3 + (-5a)x^2 + (8ab)x - 36a$$

Equating coefficients

(i) $(-4a - {}^{36}\!/_{a}) = (-5a)$

$-4a^2 - 36 = -5a^2$

$a^2 = 36$

$a = \pm 6$

(ii) $(2a^2 + 72) = 8ab$

$72 + 72 = 8ab$

$\Rightarrow ab = 18$

$a = \pm 6$

$\Rightarrow b = \pm 3$

Blunders (-3)

B1 Indices

B2 Expansion of $(x - a)^2$ once only

B3 Not like to like when equating coefficients

B4 Not 2 values of 1st variable

Attempts

A1 Other factor not linear, Att 2 marks only.

QUESTION 2

Part (a)	**15 (10, 5) marks**	**Att (3, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a)	**15 (10, 5) marks**	**Att (3, 2)**

2 (a) Solve for x: $|2x-1| \leq 3$, where $x \in \mathbf{R}$.

Limits	**10 marks**	**Att 3**
Range	**5 marks**	**Att 2**

2 (a)

$$|2x-1| \leq 3 \Rightarrow -3 \leq 2x-1 \leq 3.$$

$$\therefore\ -1 \leq x \leq 2.$$

Blunders (-3)

B1 Upper limit
B2 Lower limit
B3 Inequality sign
B4 Indices
B5 Incorrect range
B6 No range

Slips (-1)

S1 Numerical
S2 Not ≥ or ≤

Attempts

A1 Inequality sign ignored

OR

Quadratic inequality factorised	**10 marks**	**Att 3**
Range	**5 marks**	**Att 2**

2 (a)

$$|2x-1| \leq 3$$

$$(2x-1)^2 \leq 9$$

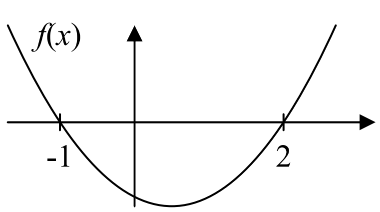
$$4x^2 - 4x + 1 \leq 9$$

$$4x^2 - 4x - 8 \leq 0$$

$$x^2 - x - 2 \leq 0$$

$$(x-2)(x+1) = 0$$

$$\Rightarrow x = 2 \quad \text{or} \quad x = -1$$



$$f(x) \leq 0$$

$$-1 \leq x \leq 2$$

Blunders (-3)

B1 Expansion of $(2x-1)^2$ once only
B2 Inequality sign
B3 Factors
B4 Root formula once only
B5 Deduction root from factor
B6 Incorrect range
B7 No range

Slips (-1)

S1 Numerical
S2 Not ≥ or ≤

Attempts

A1 Inequality signs ignored

Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

2 (b) α and $\frac{1}{\alpha}$ are roots of the quadratic equation $3kx^2-18tx+(2k+3)=0$, where t and k are constants.

(i) Find the value of k.

(ii) If one of the roots is four times the other, find the possible values of t.

(b) (i) Product of roots	**5 marks**	**Att 2**
Value k	**5 marks**	**Att 2**
(b) (ii) Value α	**5 marks**	**Att 2**
Value t	**5 marks**	**Att 2**

2 (b) (i)

$$\alpha\left(\frac{1}{\alpha}\right)=\frac{2k+3}{3k} \;\Rightarrow\; \frac{2k+3}{3k}=1 \;\Rightarrow\; k=3.$$

2 (b) (ii)

$$k=3 \Rightarrow 9x^2-18tx+9=0 \;\Rightarrow\; x^2-2tx+1=0.$$

$$\alpha=\frac{4}{\alpha} \;\Rightarrow\; \alpha^2=4 \;\Rightarrow\; \alpha=\pm 2.$$

$$\text{Sum of roots}=\alpha+\frac{1}{\alpha}=2t \;\Rightarrow\; t=\frac{1}{2}\left(\pm\frac{5}{2}\right)=\pm\frac{5}{4}.$$

Blunders (-3)

B1 Indices
B2 Sum of roots
B3 Product of roots
B4 Statement quadratic equation once only
B5 Only one value of *t,* where 2 values of α found.

Slips (-1)

S1 Numerical

Part (c) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

2 (c) Let $f(x) = \dfrac{1}{x^2 - 6x + a}$, where a is a constant.

(i) Prove that if $a = 13$, then $f(x) > 0$ for all $x \in \mathbf{R}$.

(ii) Prove that if $a = 13$, then $f(x) < 1$ for all $x \in \mathbf{R}$.

(iii) Find the range of values of a such that $0 < f(x) < 1$, for all $x \in \mathbf{R}$.

Part (c) (i) **5 marks** **Att 2**

(c) (ii) **5 marks** **Att 2**

(c) (iii) **5 marks** **Att 2**

2 (c) (i)

$$\frac{1}{x^2 - 6x + 13} = \frac{1}{(x-3)^2 + 4}.$$

$$(x-3)^2 \geq 0 \text{ for all } x \in \mathbf{R}. \quad \Rightarrow (x-3)^2 + 4 > 0.$$

$$\therefore \frac{1}{x^2 - 6x + 13} > 0 \quad \Rightarrow \quad f(x) > 0 \text{ when } a = 13.$$

2 (c) (ii)

$$\frac{1}{x^2 - 6x + 13} = \frac{1}{(x-3)^2 + 4}.$$

$$(x-3)^2 \geq 0 \Rightarrow (x-3)^2 + 4 > 1.$$

$$\therefore \frac{1}{x^2 - 6x + 13} < 1 \quad \Rightarrow \quad f(x) < 1 \text{ when } a = 13.$$

2 (c) (iii)

$$\frac{1}{x^2 - 6x + a} = \frac{1}{x^2 - 6x + 9 + (a-9)} = \frac{1}{(x-3)^2 + (a-9)}$$

So, to get $f(x)$ always > 0, we need $a > 9$, and
To get $f(x)$ always less than 1, we need denominator always > 1, so $a > 10$.
Combining these two conditions yields the overall condition $a > 10$.

Blunders (-3)

- B1 Not $(x-3)^2$
- B2 $[(x-3)^2 + 4] \not> 0$
- B3 $(x-3)^2 \not\geq 0$
- B4 $[(x-3)^2 + 4] \not> 1$
- B5 Deduction each time from work shown
- B6 No deduction each time
- B7 Inequality sign

QUESTION 3

Part (a)	**15 (10, 5) marks**	**Att (3, 2)**
Part (b)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **15 (10, 5) marks** **Att (3, 2)**

3. (a) Express $\frac{1+2i}{2-i}$ in the form of $a+bi$, where $i^2=-1$.

Multiplication by conjugate **10 marks** **Att 3**
Value **5 marks** **Att 2**

3 (a) $$\frac{1+2i}{2-i}=\frac{(1+2i)(2+i)}{(2-i)(2+i)}=\frac{2+5i+2i^2}{4-i^2}=\frac{5i}{5}=i.$$

Blunders (-3)
B1 Indices
B2 i

Slips (-1)
S1 Numerical

Attempts
A1 Not using correct conjugate

Part (b) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

3 (b) (i) Find the two complex numbers $a+bi$ such that
$$(a+bi)^2=-3+4i.$$
(ii) Hence solve the equation $x^2+x+1-i=0$.

(i) Equations **5 marks** **Att 2**
Finish **5 marks** **Att 2**
(ii) Solve **5 marks** **Att 2**

3 (b) (i)

$$(a+bi)^2=-3+4i \;\Rightarrow\; a^2-b^2+2abi=-3+4i.$$
$$\therefore a^2-b^2=-3 \text{ and } ab=2.$$
$$b=\frac{2}{a} \;\Rightarrow\; a^2-\frac{4}{a^2}=-3 \;\Rightarrow\; a^4+3a^2-4=0.$$
$$\therefore (a^2-1)(a^2+4)=0 \;\Rightarrow\; a^2-1=0 \text{ and } a^2+4\neq 0.$$
$$\therefore a=\pm 1 \;\Rightarrow\; b=\pm 2 \;\Rightarrow\; \text{solution is } \pm(1+2i).$$

3 (b) (ii)

$$x^2+x+(1-i)=0 \Rightarrow x=\frac{-1\pm\sqrt{1-4(1-i)}}{2}=\frac{-1\pm\sqrt{-3+4i}}{2}.$$

$$\therefore\ x=\frac{-1\pm(1+2i)}{2} \text{ by part (i).}$$

$$x=\frac{-1+1+2i}{2} \text{ or } x=\frac{-1-1-2i}{2} \Rightarrow x=i \text{ or } x=-1-i.$$

Blunders (-3)

B1 Expansion of $(a+ib)^2$
B2 Indices
B3 i
B4 Not like to like
B5 Factors
B6 Quadratic formula
B7 Excess values (not real)
B8 Only one complex number found
B9 Incorrect deduction root from function

Slips (-1)

S1 Answers not simplified

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

3 (c) (i) Let A and B be 2×2 matrices, where A has an inverse.

Show that $\left(A^{-1}BA\right)^n = A^{-1}B^nA$ for all $n \in \mathbb{N}$.

Let $P = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $M = \begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix}$.

(ii) Evaluate $P^{-1}MP$ and hence $\left(P^{-1}MP\right)^n$.

(iii) Hence, or otherwise, show that $M^n = M$, for all $n \in \mathbb{N}$.

Part (c) (i)	**5 marks**	**Att 2**
(c) (ii) $P^{-1}MP$	**5 marks**	**Att 2**
$(P^{-1}MP)^n$	**5 marks**	**Att 2**
(c) (iii)	**5 marks**	**Att 2**

3 (c) (i)

$$\left(A^{-1}BA\right)^n = \left(A^{-1}BA\right)\left(A^{-1}BA\right)\left(A^{-1}BA\right)\text{.............}\left(A^{-1}BA\right)$$

$$= A^{-1}B\left(AA^{-1}\right)B\left(AA^{-1}\right)\text{...........}\left(AA^{-1}\right)BA$$

$$= A^{-1}BIBI\text{............}IBA = A^{-1}BBBB\text{.........}BA$$

$$= A^{-1}B^nA.$$

(Or by induction)

3 (c) (ii)

$$P^{-1}MP = \frac{1}{(6-5)}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\left(P^{-1}MP\right)^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3 (c) (iii)

$$\left(P^{-1}MP\right)^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow P^{-1}M^nP = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\therefore\ M^n = P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}P^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix}$$

$$= M.$$

Blunders (-3)

B1 P^{-1} once only

B2 $P^{-1}.P \neq I$

B3 Indices

B4 Incorrect order of multiplication

Note: $P^{-1}MP$ must be a diagonal matrix in part (c)(ii) to merit 2nd 5 marks; otherwise 0 marks.

QUESTION 4

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**40 (5, 5, 5, 5, 10, 5, 5) marks**	**Att (2, 2, 2, 2, 3, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2, 2)**

4(a) In an arithmetic sequence, the third term is –3 and the sixth term is –15. Find the first term and the common difference.

***T*₃, *T*₆**	**5 marks**	**Att 2**
a* and *d	**5 marks**	**Att 2**

4 (a)

$$a + 2d = -3$$
$$a + 5d = -15$$
$$3d = -12 \;\Rightarrow\; d = -4 \text{ and } a = 5.$$

First term = 5, common difference = –4.

(NOTE: *a* and *d* can be in any order)

Blunders (-3)

B1 Term of arithmetic sequence
B2 Formula for term once only
B3 Incorrect *a*
B4 Incorrect *d*

Slips (-1)

S1 Numerical

Part (b)	**40 (5, 5, 5, 5, 10, 5, 5) marks**	**Att (2, 2, 2, 2, 3, 2, 2)**

4 (b) Let $u_n = l\left(\frac{1}{2}\right)^n + m(-1)^n$ for all $n \in \mathbf{N}$.

(i) Verify that u_n satisfies the equation $2u_{n+2} + u_{n+1} - u_n = 0$.

(ii) If $a_k = u_k + u_{k+1}$, express a_k in terms of *k* and *l*.

(iii) For $l > 0$, find $\sum_{k=1}^{\infty} a_k$, in terms of *l*.

(iv) Find the least positive integer *n* for which $\sum_{k=1}^{n} a_k > (0{\cdot}99)\sum_{k=1}^{\infty} a_k$.

(b) (i) Correct u_{n+1} and u_{n+2}	**5 marks**	**Att 2**
Verify	**5 marks**	**Att 2**
(b) (ii) Correct u_{k+1}	**5 marks**	**Att 2**
Express	**5 marks**	**Att 2**
(b) (iii) S_∞	**10 marks**	**Att 3**
(b) (iv) S_n	**5 marks**	**Att 2**
Least value n	**5 marks**	**Att 2**

4 (b) (i)

$$2u_{n+2}+u_{n+1}-u_n = 2l\left(\tfrac{1}{2}\right)^{n+2}+2m(-1)^{n+2}+l\left(\tfrac{1}{2}\right)^{n+1}+m(-1)^{n+1}-l\left(\tfrac{1}{2}\right)^{n}-m(-1)^{n}.$$

$$= l\left(\tfrac{1}{2}\right)^{n}\left(\tfrac{1}{2}+\tfrac{1}{2}-1\right)+m(-1)^{n}(2-1-1)=0.$$

4 (b) (ii)

$$a_k = u_k+u_{k+1} \Rightarrow a_k = l\left(\tfrac{1}{2}\right)^{k}+m(-1)^{k}+l\left(\tfrac{1}{2}\right)^{k+1}+m(-1)^{k+1}.$$

$$\therefore\ a_k = l\left(\tfrac{1}{2}\right)^{k}\left(\tfrac{3}{2}\right)+m(-1)^{k}(1-1)$$

$$= \tfrac{3}{2}l\left(\tfrac{1}{2}\right)^{k}.$$

4 (b) (iii)

$$\sum_{k=1}^{\infty} a_k = \tfrac{3}{2}l\left(\tfrac{1}{2}\right)+\tfrac{3}{2}l\left(\tfrac{1}{2}\right)^{2}+\tfrac{3}{2}l\left(\tfrac{1}{2}\right)^{3}+\text{------}+\tfrac{3}{2}l\left(\tfrac{1}{2}\right)^{k}+\ldots\ldots\ldots\ldots\ldots$$

This is an infinite geometric series. $\therefore \displaystyle\sum_{k=1}^{\infty} a_k = \frac{\tfrac{3}{4}l}{1-\tfrac{1}{2}} = \tfrac{3}{2}l.$

4 (b) (iv)

$$\sum_{k=1}^{n} a_k = \frac{\tfrac{3}{4}l\left[1-\left(\tfrac{1}{2}\right)^{n}\right]}{1-\tfrac{1}{2}} = \tfrac{3}{2}l\left[1-\left(\tfrac{1}{2}\right)^{n}\right].$$

$$\sum_{k=1}^{n} a_k > (0{\cdot}99)\sum_{k=1}^{\infty} a_k \ \Rightarrow\ \tfrac{3}{2}l\left[1-\left(\tfrac{1}{2}\right)^{n}\right] > (0{\cdot}99)\tfrac{3}{2}l\,.$$

$$\therefore\ 1-\left(\tfrac{1}{2}\right)^{n} > 0{\cdot}99 \Rightarrow \left(\tfrac{1}{2}\right)^{n} < 0{\cdot}01 \Rightarrow n=7.$$

Blunders (-3)

B1 In u_{n+1} once only
B2 In u_{n+2} once only
B3 Indices
B4 $(-1)^n$
B5 Sum of geometric progression to infinity
B6 Incorrect a
B7 Incorrect r
B8 Sum of n terms of geometric progression
B9 Not using correct values in (iv) once only
B10 Logs laws
B11 Not least integer

QUESTION 5

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

(a) Find the coefficient of x^8 in the expansion of $(x^2-1)^{10}$.

T$_7$	**5 marks**	**Att 2**
Value	**5 marks**	**Att 2**

5 (a)

$[x^2+(-1)]^{10}$ Let u_{r+1} be the rth term.

$$u_{r+1}=\binom{10}{r}(x^2)^{10-r}(-1)^r$$

$$\Rightarrow k(x^{20-2r})=k(x^8)$$
$$\Rightarrow 20-2r=8$$
$$12=2r$$
$$r=6$$

Term: $u_7=\binom{10}{6}(x^2)^4(-1)^6=\binom{10}{4}x^8=210x^8$

Coefficient: 210

OR

$$[x^2+(-1)]^{10}=(x^2)^{10}+\binom{10}{1}(x^2)^9(-1)^1+\binom{10}{2}(x^2)^8(-1)^2$$
$$+\binom{10}{3}(x^2)^7(-1)^3+\binom{10}{4}(x^2)^6(-1)^4$$
$$+\binom{10}{5}(x^2)^5(-1)^5+\binom{10}{6}(x^2)^4(-1)^6+\cdots$$

$$\Rightarrow u_7=\binom{10}{6}(x^8)(1)=210x^8$$

Coefficient: 210

Blunders (-3)

B1 General term

B2 Errors in binomial expansion once only

B3 Indices

B4 Error value $\binom{n}{r}$ or no value $\binom{n}{r}$.

Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

5 (b)

(i) Solve the equation:

$$\log_2 x - \log_2 (x-1) = 4\log_4 2 .$$

(ii) Solve the equation:

$$3^{2x+1} - 17(3)^x - 6 = 0.$$

Give your answer correct to two decimal places.

Part (b) (i) log $f(x) = 2$	**5 marks**	**Att 2**
Value x	**5 marks**	**Att 2**

5 (b) (i)

$$\log_2 x - \log_2 (x-1) = 4\log_4 2$$

$$\therefore \ \log_2 \frac{x}{x-1} = \log_4 16 = 2$$

$$\therefore \frac{x}{x-1} = 4 \ \Rightarrow \ 4x - 4 = x \ \Rightarrow \ x = \frac{4}{3}.$$

Blunders (-3)

B1 Logs laws

B2 Indices

Worthless

W1 Drops 'log'

Part (b) (ii) Quadratic factorised	**5 marks**	**Att 2**
Value x	**5 marks**	**Att 2**

5 (b) (ii)

$$3^{2x+1} - 17(3)^x - 6 = 0. \quad \text{Let } y = 3^x.$$

$$\therefore \ \ 3y^2 - 17y - 6 = 0.$$

$$(y-6)(3y+1) = 0 \quad \Rightarrow \quad y = 6, \ y \neq -\tfrac{1}{3}.$$

$$\therefore \ \ 3^x = 6 \quad \Rightarrow \quad x\log_e 3 = \log_e 6 \quad \Rightarrow \quad x = \frac{\log_e 6}{\log_e 3} = 1{\cdot}63.$$

Blunders (-3)

B1 Indices

B2 Factors once only

B3 Root formula once only

B4 Logs

B5 Uses $y = -1/3$

Slips (-1)

S1 Numerical

S2 Not to 2 decimal places

Attempts

A1 Not quadratic equation

A2 Correct answer by trial and error

Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

5 (c)

Prove by induction that 9 is a factor of $5^{2n+1} + 2^{4n+2}$, for all $n \in \mathbf{N}$.

Part (c)	***P*(1)**	**5 marks**	**Att 2**
	***P*(*k*)**	**5 marks**	**Att 2**
	***P*(*k* + 1)**	**10 marks**	**Att 3**

5 (c)

Test for $n = 1$.

$P(1)$: $5^3 + 2^6 = 189 = 9 \times 21$.

$\therefore$ True for $n = 1$.

Assume true for $n = k$.

$P(k)$: $5^{2k+1} + 2^{4k+2}$ is divisible by 9.

Test for $n = k + 1$.

$$P(k+1): \quad 5^{2k+3} + 2^{4k+6} = 25.5^{2k+1} + 16.2^{4k+2} = (9+16).5^{2k+1} + 16.2^{4k+2}$$

$$= 9.5^{2k+1} + 16\left(5^{2k+1} + 2^{4k+2}\right), \text{ which is divisible by 9.}$$

$\therefore$ True for $n = k + 1$.

So, whenever $P(k)$ is true, $P(k+1)$ true.

Since $P(1)$ true, then, by induction, $P(n)$ true for all $n \in$ N.

* Note: accept $n = 0$ as base case.

OR

5 (c)

To prove $5^{2n+1} + 2^{4n+2}$ is divisible by 9.

Test n=1

$P(1)$: $5^3 + 2^6 = 125 + 64 = 189 = 9(21)$

$\Rightarrow$ True for $n = 1$

Assume true for $n = k$

$P(k)$: $(5^{2k+1} + 2^{4k+2})$ is divisible by 9 (*)

To prove: $(5^{2k+3} + 2^{4k+6})$ is divisible by 9

Let $f(k) = 5^{2k+1} + 2^{4k+2}$

Given the assumption that $f(k)$ is divisible by 9, then $f(k+1)$ will be divisible by 9 if and only if $[f(k+1) - f(k)]$ is divisible by 9.

...ctd.

$$f(k+1)-f(k)=(5^{2k+3}+2^{4k+6})-(5^{2k+1}+2^{4k+2})$$
$$=25(5^{2k+1})+16(2^{4k+2})-5^{2k+1}-2^{4k+2}$$
$$=24(5^{2k+1})+15(2^{4k+2})$$
$$=(27-3)(5^{2k+1})+(18-3)(2^{4k+2})$$
$$=27(5^{2k+1})+18(2^{4k+2})-3(5^{2k+1})-3(2^{4k+2})$$
$$=\underbrace{9[3(5^{2k+1})+2(2^{4k+2})]}_{\text{Divisible by 9}}-\underbrace{3[5^{2k+1}+2^{4k+2}]}_{\text{Divisible by 9 from (*) above}}$$

$\Rightarrow f(k+1)-f(k)$ is divisible by 9

So whenever *P*(*k*) true, *P*(*k*+1) is true. Since *P*(1) is true, then by induction *P*(2), *P*(3), *P*(4)……. are all true.

Blunders (-3)
B1 Indices
B2 $n \geq 2$

Slips (-1)
S1 Numerical

Note: Must prove *P*(1) step. Not sufficient to state *P*(*n*) true for *n*=1

QUESTION 6

Part (a)	**15 marks**	**Att 5**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a) **15 marks** **Att 5**

(a) Differentiate $\cos^2 x$ with respect to x.

6 (a)

$$f(x) = \cos^2 x \quad \Rightarrow \quad f'(x) = -2\cos x \sin x.$$

Blunders (-3)

B1 Differentiation

Attempts

A1 Error in differentiation formula (chain rule)

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

6 (b) The equation of a curve is $y = e^{-x^2}$.

(i) Find $\frac{dy}{dx}$.

(ii) Find the co-ordinates of the turning point of the curve.

(iii) Determine whether this turning point is a local maximum or a local minimum.

Part (b) (i)	**5 marks**	**Att 2**
(ii) $f'(x) = 0$	**5 marks**	**Att 2**
Turning point	**5 marks**	**Att 2**
(iii)	**5 marks**	**Att 2**

6 (b) (i)

$$\frac{dy}{dx} = e^{-x^2}(-2x).$$

6 (b) (ii)

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad e^{-x^2}(-2x) = 0 \quad \Rightarrow \quad x = 0 \text{ and } y = 1. \text{ Turning point is } (0,1).$$

6 (b) (iii)

$$\frac{d^2y}{dx^2} = e^{-x^2}(-2x)(-2x) - 2e^{-x^2} = e^{-x^2}(4x^2 - 2).$$

$$\text{For } x = 0, \ \frac{d^2y}{dx^2} = -2e^0 = -2 < 0 \ \Rightarrow \ (0,1) \text{ is a local maximum.}$$

Blunders (-3)

B1 Indices

B2 Differentiation

B3 $e^{-x^2} = 0$

B4 No 2nd differential

Attempts

A1 Error in differentiation formula (chain rule)

Note: Over simplified work in (i) can lead to attempt at most in (ii) and (iii).

Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

6 (c) The function *f* is defined as $x \rightarrow \dfrac{2x}{x+1}$, where $x \in \mathbf{R} \backslash \{-1\}$.

(i) Find the equations of the asymptotes of the curve $y = f(x)$.

(ii) *P* and *Q* are distinct points on the curve $y = f\{x)$. The tangent at *Q* is parallel to the tangent at *P*. The co-ordinates of *P* are $(1, 1)$.
Find the co-ordinates of *Q*.

(iii) Verify that the point of intersection of the asymptotes is the midpoint of $[PQ]$.

Part (c) (i)	**5 marks**	**Att 2**
(ii)	**5 marks**	**Att 2**
(iii)	**5 marks**	**Att 2**

6 (c) (i)

$x = -1$ is the vertical asymptote.

$$\lim_{x\to\infty}\frac{2x}{x+1} = \lim_{x\to\infty}\frac{2}{1+\frac{1}{x}} = 2 \quad \Rightarrow \quad y = 2 \text{ is a horizontal asymptote.}$$

6 (c) (ii)

$$f'(x) = \frac{2(x+1) - 2x(1)}{(x+1)^2} = \frac{2}{(x+1)^2}. \quad \text{Slope at } P(1,1) = \frac{2}{4} = \frac{1}{2}.$$

$$\text{Slope at } Q = \frac{1}{2} \Rightarrow \frac{2}{(x+1)^2} = \frac{1}{2} \Rightarrow (x+1)^2 = 4.$$

$$\therefore\ x + 1 = \pm 2 \Rightarrow x = 1 \text{ or } x = -3. \quad \therefore\ Q \text{ is } (-3, 3).$$

OR

$$(x+1)^2 = 4$$
$$x^2 + 2x + 1 - 4 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$\Rightarrow x+3=0 \quad \text{or} \quad x-1=0$$
$$x=-3 \quad \text{or} \quad \text{x}=1$$
$$\downarrow \qquad\qquad \downarrow$$
$$Q(-3,3) \qquad\qquad P(1,1)$$

6 (c) (iii) Asymptotes intersect at $(-1, 2)$,
$P(1,1)$ and $Q(-3,3)$.
Mid-point of $[PQ]$ is $(-1,2)$.

Blunders (-3)
B1 Asymptotes
B2 Limits
B3 Differentiation
B4 Indices
B5 Formula for mid-point line

Slips (-1)
S1 Numerical

Attempts
A1 Error in differentiation formula

Note: Cannot get 2nd 5 marks in (c) (ii) if slope at *Q* not equal to slope at *P*.

QUESTION 7

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**25 (10, 10, 5) marks**	**Att (3, 3, 2)**
Part (c)	**15 (10, 5) marks**	**Att (3, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

7 (a) Find the slope of the tangent to the curve $x^2 + y^3 = x - 2$ at the point (3, –2).

Differentiation	**5 marks**	**Att 2**
Slope	**5 marks**	**Att 2**

7 (a)

$$2x + 3y^2 \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1-2x}{3y^2}. \quad \therefore \text{ Slope of tangent at } (3,-2) = \frac{-5}{12}.$$

Blunders (-3)

B1 Differentiation

B2 Indices

B3 Incorrect value of x or no value of x in slope

B4 Incorrect value of y or no value of y in slope

Slips (-1)

S1 Numerical

Attempts

A1 Error in differentiation formula

A2 $\frac{dy}{dx} = 2x + 3y^2 \frac{dy}{dx} = 1$ and uses the two $\left(\frac{dy}{dx}\right)$ terms

Part (b) **25 (10, 10, 5) marks** **Att (3, 3, 2)**

7 (b) A curve is defined by the parametric equations

$$x = \frac{t-1}{t+1} \quad \text{and} \quad y = \frac{-4t}{(t+1)^2}, \text{ where } t \neq -1.$$

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

(ii) Hence find $\frac{dy}{dx}$, and express your answer in terms of x.

Part (b) $\boldsymbol{\frac{dx}{dt}}$ **10 marks** **Att 3**

$\boldsymbol{\frac{dy}{dt}}$ **10 marks** **Att 3**

$\boldsymbol{\frac{dy}{dx}}$ **5 marks** **Att 2**

7 (b) (i)

$$\frac{dx}{dt} = \frac{1(t+1)-1(t-1)}{(t+1)^2} = \frac{2}{(t+1)^2}.$$

$$\frac{dy}{dt} = \frac{-4(t+1)^2 + 4t(2)(t+1)}{(t+1)^4} = \frac{-4(t+1)+8t}{(t+1)^3} = \frac{4(t-1)}{(t+1)^3}.$$

7 (b) (ii)

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{4(t-1)}{(t+1)^3} \times \frac{(t+1)^2}{2} = \frac{2(t-1)}{t+1} = 2x.$$

Blunders (-3)

B1 Differentiation
B2 Indices
B3 Error in getting $\frac{dy}{dx}$

Attempts

A1 Error in differentiation formula

Part (c) **15 (10, 5) marks** **Att (3, 2)**

(c) The functions f and g are defined on the domain $\mathbb{R}\backslash\{-1, 0\}$ as follows:

$$f : x \to \tan^{-1}\left(\frac{-x}{x+1}\right) \quad \text{and} \quad g : x \to \tan^{-1}\left(\frac{x+1}{x}\right).$$

(i) Show that $f'(x) = \dfrac{-1}{2x^2+2x+1}$.

(ii) It can be shown that $f'(x) = g'(x)$.

One of the three diagrams A, B, or C below represents parts of the graphs of f and g. Based only on the derivatives, state which diagram is the correct one, and state also why each of the other two diagrams is incorrect.

Diagram *A*

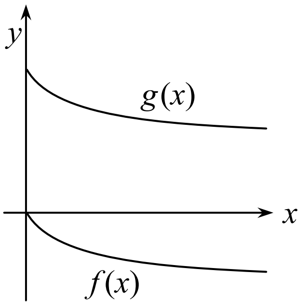


Diagram *B*

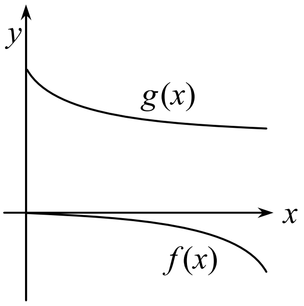
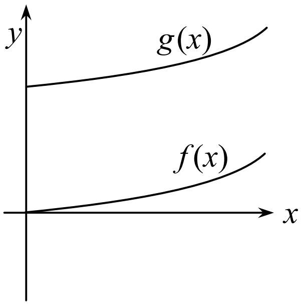


Diagram *C*



c(i) **10 marks** **Att 3**

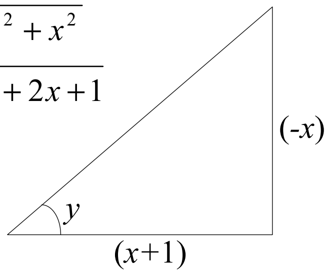
7 (c) (i)

$$f(x) : x \to \tan^{-1}\left(\frac{-x}{x+1}\right)$$

$$f'(x) = \frac{1}{1+\left(\frac{-x}{x+1}\right)^2} \times \frac{-1(x+1)+x(1)}{(x+1)^2} = \frac{(x+1)^2}{x^2+2x+1+x^2} \times \frac{-1}{(x+1)^2} = \frac{-1}{2x^2+2x+1}.$$

OR

$$\sqrt{(x+1)^2+x^2}$$
$$= \sqrt{2x^2+2x+1}$$



$$y = \tan^{-1}\left(\frac{-x}{x+1}\right)$$

$$\tan y = \frac{-x}{x+1}$$

$$\sec^2 y . \frac{dy}{dx} = \frac{(x+1)(-1)-(-x)(1)}{(x+1)^2}$$

$$\frac{1}{\cos^2 y} . \frac{dy}{dx} = \frac{-x-1+x}{(x+1)^2}$$

$$\frac{1}{\cos^2 y} . \frac{dy}{dx} = \frac{-1}{(x+1)^2}$$

$$\cos y = \frac{x+1}{\sqrt{2x^2+2x+1}}$$

$$\cos^2 y = \frac{(x+1)^2}{2x^2+2x+1}$$

(…/)

$$\frac{dy}{dx}=\frac{-\cos^2 y}{(x+1)^2}$$
$$=\frac{-1}{(x+1)^2}.\frac{(x+1)^2}{2x^2+2x+1}$$
$$=\frac{-1}{2x^2+2x+1}$$

Blunders (-3)
B1 Differentiation
B2 Indices
B3 Error in value of tan y
B4 Error in value of cos y
B5 Sides of triangle once only

Attempts
A1 Error in differentiation formula and hence Att 2 at most in simplification

Part (c) (ii) **5 marks** **Att 2**

7 (c) (ii)

Diagram *A* is correct.

It cannot be Diagram *B*, as these curves are not "parallel" (i.e. identical up to a vertical shift, which is necessary because their derivatives are equal for all x).

It cannot be Diagram *C* as these graphs are increasing, whereas they should be decreasing, because their derivatives are negative for $x > 0$.

OR

$$\text{Given } f'(x)=g'(x)$$
$$\Rightarrow m_1=m_2 \qquad \text{(same slopes)}$$
$$\Rightarrow \text{parallel curves}$$
$$f'(x)=\frac{-1}{2x^2+2x+1}<0 \quad \text{when } x>0$$
$$\Rightarrow \text{Both } f(x) \text{ and } g(x) \text{ are decreasing funtions.}$$

Diagram A: correct
Diagram B: not parallel curves
Diagram C: increasing curves

Blunders (-3)
B1 Incorrect statement

QUESTION 8

Part (a)	**15 marks**	**Att 5**
Part (b)	**25 (5, 5, 5, 5, 5) marks**	**Att (2, 2, 2, 2, 2)**
Part (c)	**10 (5, 5) marks**	**Att (2, 2)**

Part (a) **15 marks** **Att 5**

8 **(a)** Find $\int\left(x^3+\sqrt{x}\right)dx$.

8 (a)

$$\int\left(x^3+\sqrt{x}\right)\,dx \;=\; \tfrac{1}{4}x^4+\tfrac{2}{3}x^{\frac{3}{2}}+c.$$

Blunders (-3)

B1 Integration
B2 Indices
B3 No ‘c’

Part (b) **25 (5, 5, 5, 5, 5) marks** **Att (2, 2, 2, 2, 2)**

8 (b) **(i)** Evaluate $\int_0^2 \frac{x+1}{x^2+2x+2}\,dx$.

(ii) Evaluate $\int_0^2 \frac{x^2+2x+2}{x+1}\,dx$.

Part (b) (i) Correct substitution	**5 marks**	**Att 2**
Integration	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

8 (b) (i)

$$\int_0^2 \frac{x+1}{x^2+2x+2}\,dx. \qquad \text{Let } u=x^2+2x+2 \;\Rightarrow\; du=(2x+2)dx.$$

$$=\frac{1}{2}\int_2^{10}\frac{du}{u}=\tfrac{1}{2}\left[\log_e u\right]_2^{10}=\tfrac{1}{2}\left[\log_e 10-\log_e 2\right]=\tfrac{1}{2}\log_e 5=\log_e\sqrt{5}.$$

Blunders (-3)

B1 Integration
B2 Differentiation
B3 Logs
B4 Limits
B5 Incorrect order in applying limits

B6 Not calculating substituted limits
B7 Not changing limits

Slips (-1)
S1 Numerical

Part (b) (ii) Integration **5 marks** **Att 2**
Finish **5 marks** **Att 2**

8 (b) (ii)

$$\therefore \int_0^2 \frac{x^2+2x+2}{x+1}\,dx = \int_0^2 \frac{(x+1)^2+1}{x+1}dx = \int_0^2 \left((x+1)+\frac{1}{x+1}\right)dx$$

$$= \left[\tfrac{1}{2}x^2+x+\log_e(x+1)\right]_0^2 = 2+2+\log_e 3 = 4+\log_e 3.$$

OR

$$\begin{array}{r} x+1 \\ x+1\overline{)x^2+2x+2} \\ \underline{x^2+x} \\ x+2 \\ \underline{x+1} \\ 1 \end{array}$$

$$\int \frac{x^2+2x+2}{x+1}dx$$

$$= \int [(x+1)+(\frac{1}{x+1})]dx$$

Finish as above

Blunders (-3)
B1 Integration
B2 Differentiation
B3 Logs
B4 Limits
B5 Incorrect order in applying limits
B6 Not calculating substituted limits
B7 Not changing limits

Slips (-1)
S1 Numerical
S2 Not changing sign when subtracting in division

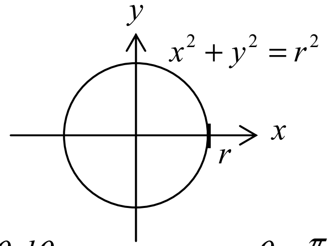
Part (c) **10 (5, 5) marks** **Att (2, 2)**

8 (c) Use integration methods to establish the formula $A = \pi r^2$ for the area of a disc of radius *r*.

Set up **5 marks** **Att 2**
Finish **5 marks** **Att 2**

8 (c)

$x^2 + y^2 = r^2$ is a circle, centre $(0, 0)$, radius $= r$.



Area of disc $= A = 4\int_0^r \sqrt{r^2 - x^2}\,dx$

Let $x = r\sin\theta \Rightarrow \quad dx = r\cos d\theta.$

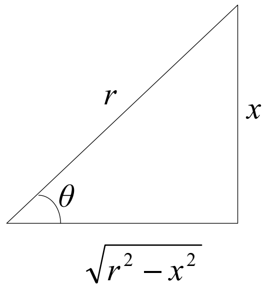
$$\therefore A = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}.r\cos\theta\, d\theta = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2\left(1 - \sin^2\theta\right)}.r\cos\theta\, d\theta \qquad x = r \Rightarrow \theta = \pi/2$$

$$= 4\int_0^{\frac{\pi}{2}} r^2\cos^2\theta\, d\theta = \left(4r^2\right)\frac{1}{2}\left[\theta + 1/2\sin 2\theta\right]_0^{\frac{\pi}{2}} \qquad x = 0 \Rightarrow \theta = 0$$

$$= 2r^2\left[\left(\frac{\pi}{2} + \sin\pi\right) - \left(0 + \sin 0\right)\right]$$

$$\therefore A = 2r^2\left(\frac{\pi}{2}\right) = \pi r^2.$$

OR



$$\frac{x}{r} = \sin\theta \Rightarrow x = r\sin\theta$$

$$\frac{dx}{d\theta} = r\cos\theta \Rightarrow dx = r\cos\theta.d\theta$$

$$\text{From diagram: } \cos\theta = \frac{\sqrt{r^2 - x^2}}{r} \Rightarrow r\cos\theta = \sqrt{r^2 - x^2}$$

$$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx$$

$$= 4\int (r\cos\theta).(r\cos\theta).d\theta$$

$$= 4\int r^2\cos^2\theta.d\theta \quad \text{etc.}$$

Blunders (-3)
B1 Integration
B2 Differentiation
B3 Trig formula
B4 Indices
B5 Limits
B6 Incorrect order in applying limits

B7 Not calculating substituted limits
B8 Not changing limits
B9 Definition of $\sin\theta$
B10 Definition of $\cos\theta$

Slips (-1)
S1 Numerical
S2 Trig value or no trig value

Attempts
A1 Error in differentiation formula or rules of integration

Worthless
W1 $x = r\sin\theta$ or $x = r\cos\theta$ not used in integration: 0 marks for 2nd 5

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate 2011

Marking Scheme

Mathematics – Paper 2

Higher Level

GENERAL GUIDELINES FOR EXAMINERS – PAPER 2

1. Penalties of three types are applied to candidates' work as follows:
   - Blunders - mathematical errors/omissions (-3)
   - Slips - numerical errors (-1)
   - Misreadings (provided task is not oversimplified) (-1).

   Frequently occurring errors to which these penalties must be applied are listed in the scheme. They are labelled: B1, B2, B3,…, S1, S2,…, M1, M2,…etc. These lists are not exhaustive.

2. When awarding attempt marks, e.g. Att(3), note that
   - any *correct, relevant* step in a part of a question merits at least the attempt mark for that part
   - if deductions result in a mark which is lower than the attempt mark, then the attempt mark must be awarded
   - a mark between zero and the attempt mark is never awarded.

3. Worthless work is awarded zero marks. Some examples of such work are listed in the scheme and they are labelled as W1, W2, …etc.

4. The phrase "hit or miss" means that partial marks are not awarded – the candidate receives all of the relevant marks or none.

5. The phrase "and stops" means that no more work of merit is shown by the candidate.

6. Special notes relating to the marking of a particular part of a question are indicated by an asterisk. These notes immediately follow the box containing the relevant solution.

7. The sample solutions for each question are not intended to be exhaustive lists – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

8. Unless otherwise indicated in the scheme, accept the best of two or more attempts – even when attempts have been cancelled.

9. The *same* error in the *same* section of a question is penalised *once* only.

10. Particular cases, verifications and answers derived from diagrams (unless requested) qualify for attempt marks at most.

11. A serious blunder, omission or misreading results in the attempt mark at most.

12. Do not penalise the use of a comma for a decimal point, e.g. €5·50 may be written as €5,50.

QUESTION 1

Part (a)	**10 marks**	**Att 3**
Part (b)	**25 (10, 5, 5, 5) marks**	**Att (3, 2, 2, 2)**
Part (c)	**15 (10, 5) marks**	**Att (3, 2)**

Part (a) **10 marks** **Att 3**

1 (a) The following parametric equations define a circle:

$$x = 2 + 3\sin\theta, \quad y = 3\cos\theta, \text{ where } \theta \in \mathbf{R}.$$

What is the Cartesian equation of the circle?

Part (a) **10 marks** **Att 3**

1 (a)

$$x = 2 + 3\sin\theta \qquad y = 3\cos\theta$$

$$(x-2)^2 + y^2 = 9\sin^2\theta + 9\cos^2\theta = 9(\cos^2\theta + \sin^2\theta).$$

$$\therefore\ (x-2)^2 + y^2 = 9.$$

OR

$$x^2 = 4 + 12\ \sin\theta + 9\sin^2\theta \quad \text{and} \quad y^2 = 9\cos^2\theta$$

$$\Rightarrow x^2 + y^2 = 4 + 12\sin\theta + 9(\sin^2\theta + \cos^2\theta) = 13 + 12\sin\theta$$

$$\Rightarrow x^2 + y^2 = 13 + 12(\frac{x-2}{3})$$

$$\Rightarrow x^2 + y^2 = 13 + 4x - 8$$

$$\Rightarrow x^2 + y^2 - 4x - 5 = 0$$

OR

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \qquad \Rightarrow (x-2)^2 + y^2 = 9$$

OR

Centre (2,0) and Radius 3 $\Rightarrow (x-2)^2 + y^2 = 9$

Blunders (-3)

B1 Incorrect squaring (apply once if same type of error)

B2 $\cos^2\theta + \sin^2\theta \neq 1$

B3 Incorrect centre or radius

Slips (-1)

S1 Arithmetic error

Attempts (3 marks)

A1 Effort at expressing x^2 or y^2 in terms of θ

A2 θ not eliminated

A3 Centre (2,0) and/or radius 3 and stops

A4 $x^2 + y^2 = 9$ with work

Worthless

W1 $x^2 + y^2 = 1$

Part (b)	**25 (10, 5, 5, 5) marks**	**Att (3, 2, 2, 2)**

1 (b) Find the equation of the circle that passes through the points (0, 3), (2, 1) and (6, 5).

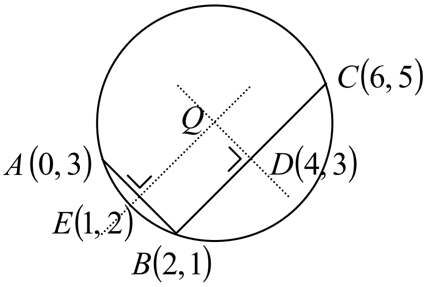
(b) One mediator	**10 marks**	**Att 3**
2nd mediator	**5 marks**	**Att 2**
Centre	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1 (b)

Mid-point $[AB] = E(1, 2)$.

Slope of $AB = \dfrac{3-1}{0-2} = -1 \Rightarrow$ slope $EQ = 1$.

$\therefore$ Equation $EQ: y - 2 = 1(x-1) \Rightarrow EQ: x - y = -1$.



Mid-point $[BC] = D(4, 3)$.

Slope of $BC = \dfrac{5-1}{6-2} = 1 \Rightarrow$ slope of $DQ = -1$.

$\therefore$ Equation $DQ: y - 3 = -1(x-4) \Rightarrow DQ: x + y = 7$.

$x - y = -1$

$\underline{x + y = 7}$

$2x = 6 \Rightarrow x = 3$ and $y = 4$. $\therefore$ Centre Q is $(3, 4)$.

$|AQ| = r = \sqrt{(3-0)^2 + (4-3)^2} = \sqrt{10}$. Equation of circle: $(x-3)^2 + (y-4)^2 = 10$.

OR

(b) An equation in two variables	**10 marks**	**Att 3**
Second equation in two variables	**5 marks**	**Att 2**
Two values	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1(b)

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 0 + 9 + 2g(0) + 2f(3) + c = 0 \Rightarrow 6f + c = -9$.

Also $4 + 1 + 4g + 2f + c = 0 \Rightarrow 4g + 2f + c = -5$ ....(i)

and $36+25+12g+10f+c=0 \Rightarrow 12g + 10f + c = -61$ .....(ii)

Solving between (i) and (ii) $g = -3$ and $f = -4$

$\Rightarrow 6(-4) + c = -9 \Rightarrow c = 15$

Equation of circle: $x^2 + y^2 - 6x - 8y + 15 = 0$

OR

(b) Appropriate slopes	**10 marks**	**Att 3**
Establishing semi circle	**5 marks**	**Att 2**
Centre or radius	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1(b) Slope (0,3) and (2,1) $\frac{1-3}{2-0}=-1$

Slope (2,1) and (6,5) $\frac{5-1}{6-2}=1$

$\Rightarrow$ perpendicular lines.

But angle in a semi-circle right angle $\Rightarrow$ (0,3) and (6,5) diameter extremities.

Centre of circle (3,4)

Radius: $\sqrt{(3-0)^2+(4-3)^2}=\sqrt{10}$

Equation: $(x-3)^2+(y-4)^2=10$

Blunders (-3)

B1 Incorrect perpendicular slope
B2 Error in slope formula
B3 Error in equation of line formula
B4 Error in radius formula
B5 Equation of circle incomplete
B6 Incorrect diameter
B7 Error in general equation of circle
B8 Equation of circle but radius not calculated

Slips (-1)

S1 Arithmetic errors

Attempts (3, 2, 2, 2 marks)

A1 Product of perpendicular slopes = -1
A2 Mixing x and y ordinates
A3 Correct formula with some correct substitution
A4 Some correct substitution into general equation of circle

Part (c)	**15 (10, 5) marks**	**Att (3, 2)**

1 (c) The circle c_1: $x^2+y^2-8x+2y-23=0$ has centre A and radius r_1.

The circle $c_2: x^2+y^2+6x+4y+3=0$ has centre B and radius r_2.

(i) Show that c_1 and c_2 intersect at two points.

(ii) Show that the tangents to c_1 at these points of intersection pass through the centre of c_2.

Part (c)(i) **10 marks** **Att 3**

1 (c) (i)

$A(4,-1)$ and $r_1 = \sqrt{16+1+23} = \sqrt{40} = 2\sqrt{10}.$

$B(-3,\ -2)$ and $r_2 = \sqrt{9+4-3} = \sqrt{10}.$

$|AB| = \sqrt{(4+3)^2 + (-1+2)^2} = \sqrt{50} = 5\sqrt{2}.$

So, $r_1 + r_2 = 3\sqrt{10} = \sqrt{90} > \sqrt{50}$ and $|r_1 - r_2| = \sqrt{10} < \sqrt{50}$

$\Rightarrow$ circles intersect at two points.

OR

Part (c)(i) **10 marks** **Att 3**

1(c)(i)

$$\begin{array}{l} x^2 + y^2 - 8x + 2y - 23 = 0 \\ \underline{x^2 + y^2 + 6x + 4y + 3 = 0} \end{array}$$

$$-14x - 2y - 26 = 0 \Rightarrow y = -7x - 13$$

$$x^2 + (-7x-13)^2 - 8x + 2(-7x-13) - 23 = 0$$

$$\Rightarrow 5x^2 + 16x + 12 = 0$$

$$\Rightarrow (5x+6)(x+2) = 0$$

$$\Rightarrow x = \frac{-6}{5}, x = -2$$

$$\Rightarrow y = \frac{-23}{5}, y = 1$$

Two points of intersection $\left(\frac{-6}{5}, \frac{-23}{5}\right)$ and (-2,1)

Blunders (-3)

B1 Relationship between $3\sqrt{10}$ and $\sqrt{50}$ or $\sqrt{40} + \sqrt{10} > \sqrt{50}$ not clearly established
B2 Error in squaring
B3 Error in factors
B4 Incorrect conclusion stated or implied

Slips (-1)

S1 Arithmetic errors
S2 Not establishing both cases

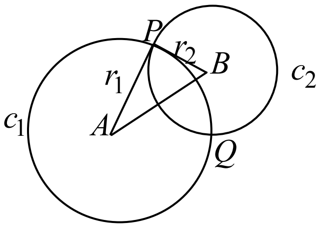
Attempts (3 marks)

A1 One centre and radius found
A2 Expressing *y* in terms of *x* and stops

Part (c) (ii) 5 marks Att 2

1 (c) (ii)

Let P and Q be the points of intersection of the circles.
The tangent to c_1 passes through B, if and only if APB and AQB are right-angled triangles.



$$|AP|^2 + |BP|^2 = r_1{}^2 + r_2{}^2 = 40 + 10 = 50 = |AB|^2.$$

$$\therefore |\angle APB| = 90^\circ \Rightarrow AP \perp PB.$$

$\therefore$ PB is a tangent to c_1 and contains centre B of c_2.

Similarly QB is a tangent to c_1 and contains centre B of c_2.

OR

Part (c)(ii) 5 marks Att 2

1(c)(ii)

Slope diameter: centre(4,-1) and point of contact (-2,1)

$$\frac{-1-1}{4+2} = \frac{-1}{3} \Rightarrow \text{slope of tangent equals 3}$$

Equation of tangent: y-1=3(x+2) $\Rightarrow$ 3x-y+7=0

But (-3,-2) lies on tangent since 3(-3)-1(-2)+7 = -9+2+7=0

Slope (4,-1) and $\left(\frac{-6}{5}, \frac{-23}{5}\right)$ equals $\frac{9}{13}$ $\Rightarrow$ slope of tangent equals $\frac{-13}{9}$

Equation of tangent: $y + \frac{23}{5} = \frac{-13}{9}\left(x + \frac{6}{5}\right)$

But (-3,-2) lies on this tangent since

$$\text{LHS: } -2 + \frac{23}{5} = \frac{13}{5} \quad \text{and} \quad \text{RHS: } \frac{-13}{9}\left(-3 + \frac{6}{5}\right) = \frac{-13}{9}\left(\frac{-9}{5}\right) = \frac{13}{5}$$

Blunders (-3)

B1 Incorrect use of Pythagoras
B2 One case only
B3 Incorrect slope or equation of line formula with substitution
B4 Not verifying centre on tangents

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 Squaring one radius and stops
A2 Equation of one tangent only and stops

Misreading(-1)

M1 Centres interchanged

QUESTION 2

Part (a)	**15 marks**	**Att 5**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a) **15 marks** **Att 5**

2 (a) Find the value of s and the value of t that satisfy the equation

$$s(\vec{i}-4\vec{j})+t(2\vec{i}+3\vec{j})=4\vec{i}-27\vec{j}\,.$$

Part (a) **15 marks** **Att 5**

2 (a)

$$s(\vec{i}-4\vec{j})+t(2\vec{i}+3\vec{j})=4\vec{i}-27\vec{j}$$

$$\therefore\ \vec{i}(s+2t)+\vec{j}(-4s+3t)=4\vec{i}-27\vec{j}.$$

$$s+2t=4 \qquad \Rightarrow\ 4s+8t=16$$

$$-4s+3t=-27 \qquad \underline{-4s+3t=-27}$$

$$11t=-11\ \Rightarrow\ t=-1 \text{ and } s=6.$$

Blunders (-3)

B1 One value only

Slips (-1)

S1 Arithmetic errors

Attempts (5 marks)

A1 One equation in s and t

Part (b) **20 (10, 10) marks** **Att (3, 3)**

2 (b) $\overrightarrow{OP}=3\vec{i}-4\vec{j}$ and $\overrightarrow{OQ}=5\left(\overrightarrow{OP}^{\perp}\right)$.

(i) Find $\overrightarrow{OQ}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find $\cos|\angle OQP|$, in surd form.

Part (b) (i) **10 marks** **Att 3**

2 (b) (i)

$$\overrightarrow{OP}=3\vec{i}-4\vec{j}\ \Rightarrow\ \overrightarrow{OP}^{\perp}=4\vec{i}+3\vec{j}.$$

$$\therefore\ \overrightarrow{OQ}=20\vec{i}+15\vec{j}.$$

Blunders (-3)

B1 Error in $\overrightarrow{OP}^{\perp}$

B2 $\overrightarrow{OQ}=\left(\overrightarrow{OP}^{\perp}\right)$

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)

A1 Relationship between a vector and related perpendicular stated or implied

Part (b) (ii) **10 marks** **Att 3**

2 (b) (ii)

$$\cos\angle OQP = \frac{\left(\overrightarrow{OQ}\right)\cdot\left(\overrightarrow{PQ}\right)}{\left|\overrightarrow{OQ}\right|\left|\overrightarrow{PQ}\right|} = \frac{\left(20\vec{i}+15\vec{j}\right)\left(17\vec{i}+19\vec{j}\right)}{\left|20\vec{i}+15\vec{j}\right|\left|17\vec{i}+19\vec{j}\right|}$$

$$= \frac{340+285}{\sqrt{400+225}\sqrt{289+461}} = \frac{625}{\sqrt{625}\sqrt{650}} = \frac{25}{5\sqrt{26}} = \frac{5}{\sqrt{26}}.$$

Blunders (-3)

B1 $\overrightarrow{PQ} \neq \vec{q} - \vec{p}$

B2 Error in modulus formula

B3 Answer not in single surd

Slips (-1)

S1 Arithmetic errors

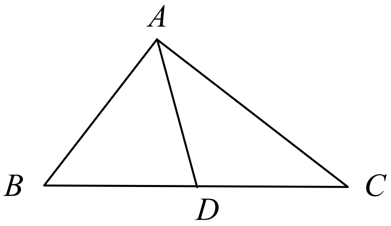
Attempts (3 marks)

A1 $\cos\angle POQ$ calculated

A2 $\cos\,\theta$ formula with some correct substitution

Part (c) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

2 **(c)** ABC is a triangle and D is the mid-point of $[BC]$.



(i) Express $\overrightarrow{AB}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$ and express $\overrightarrow{AC}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

(ii) Hence, prove that $|AB|^2 + |AC|^2 = 2|AD|^2 + \frac{1}{2}|BC|^2$.

Part (c) (i) **10 (5, 5) marks** **Att (2, 2)**

2 (c) (i)

$$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{BC}.$$

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{BC}.$$

Blunders (-3)

B1 $\overrightarrow{DB} \neq -\frac{1}{2}\overrightarrow{BC}$

B2 $\overrightarrow{DC} \neq \frac{1}{2}\overrightarrow{BC}$

Attempts (2, 2marks)

A1 $\overrightarrow{AB}$ and/or $\overrightarrow{AC}$ as the sum of two vectors

Part (c) (ii) **5 marks** **Att 2**

2 (c) (ii)

$$|AB|^2 = \overrightarrow{AB}.\overrightarrow{AB} = \left(\overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{BC}\right)\left(\overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{BC}\right) = |AD|^2 + \tfrac{1}{4}|BC|^2 - \tfrac{1}{2}\overrightarrow{AD}.\overrightarrow{BC} - \tfrac{1}{2}\overrightarrow{BC}.\overrightarrow{AD}$$

$$|AC|^2 = \overrightarrow{AC}.\overrightarrow{AC} = \left(\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{BC}\right)\left(\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{BC}\right) = |AD|^2 + \tfrac{1}{4}|BC|^2 + \tfrac{1}{2}\overrightarrow{AD}.\mathrm{BC} + \tfrac{1}{2}\overrightarrow{BC}.\overrightarrow{AD}$$

$$\therefore |AB|^2 + |AC|^2 = 2|AD|^2 + \tfrac{1}{2}|BC|^2.$$

Blunders (-3)

B1 Incorrect conclusion or no conclusion implied

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 $\left(\overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{BC}\right)\left(\overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{BC}\right) = |AD|^2 + \tfrac{1}{4}|BC|^2$

A2 $|AB|^2$ or $\left(\overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{BC}\right)\left(\overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{BC}\right) = |AD|^2 + \tfrac{1}{4}|BC|^2 - \overrightarrow{AD}\cdot\overrightarrow{BC}$

A3 $\overrightarrow{AB}\cdot\overrightarrow{AB} = \overrightarrow{AB}^2$ or $|AB|^2$

Worthless (0 marks)

W1 $|AB|^2 = |AD|^2 + \tfrac{1}{4}|BC|^2$

QUESTION 3

Part (a)	**15 marks**	**Att 5**
Part (b)	**35 (20, 5, 5, 5) marks**	**Att (7, 2, 2, 2)**

Part (a) **15 marks** **Att 5**

3 (a) *P* and *Q* are the points (–1, 4) and (3, 7) respectively.
Find the co-ordinates of the point that divides $[PQ]$ internally in the ratio 3 : 1.

Part (a) **15 marks** **Att 5**

3 (a)

Point is $\left(\dfrac{1(-1)+3(3)}{3+1}, \dfrac{1(4)+3(7)}{3+1}\right) = \left(\dfrac{8}{4}, \dfrac{25}{4}\right) = \left(2, 6\tfrac{1}{4}\right).$

*Note: General Guideline 8 does not necessarily apply here

Blunders (-3)
B1 Incorrect ratio formula
B2 Incorrect translation

Slips (-1)
S1 Arithmetic errors

Attempts (5 marks)
A1 One correct ordinate

Worthless (0 marks)
W1 Midpoint used once

Part (b) **35 (20, 5, 5, 5) marks** **Att (7, 2, 2, 2)**

3 (b) *f* is the transformation $(x, y) \to (x', y')$, where $x' = x - y$ and $y' = 2x + 3y$.
l_1 is the line $2x - y - 1 = 0$.

(i) Find the equation of $f(l_1)$, the image of l_1 under *f*.

(ii) Prove that *f* maps every pair of parallel lines to a pair of parallel lines.
You may assume that *f* maps every line to a line.

(iii) The line l_2 is parallel to the line l_1.
$f(l_2)$ intersects the *x*-axis at A' and the *y*-axis at B'.
The area of the triangle $A'OB'$ is 9 square units, where *O* is the origin.
Find the two possible equations of l_2.

(iv) Given that $A' = f(A)$ and $B' = f(B)$, show that $|\angle AOB| \neq |\angle A'OB'|$.

Part (b)(i) **20 marks** **Att 7**

3 (b) (i)

$$\begin{array}{l} 2x' = 2x - 2y \\ \underline{y' = 2x + 3y} \end{array}$$

$$2x' - y' = -5y \;\Rightarrow\; y = \frac{1}{5}\left(-2x' + y'\right)$$

$$x = x' + y \;\Rightarrow\; x = x' + \frac{1}{5}\left(-2x' + y'\right) \;\Rightarrow\; x = \frac{1}{5}\left(3x' + y'\right).$$

$$f\left(l_1\right): \frac{2}{5}\left(3x' + y'\right) - \frac{1}{5}\left(-2x' + y'\right) - 1 = 0 \;\Rightarrow\; 8x' + y' - 5 = 0.$$

Blunders (-3)

B1 $f(l_1)$ not in form $px' + qy' + r = 0$ or $y' = mx' + c$

B2 Incorrect matrix

B3 Incorrect matrix multiplication

Slips (-1)

S1 Arithmetic errors

Attempts (7 marks)

A1 Effort at x or y expressed in terms of x' and y'

A2 Correct matrix for f when finding $f(l_1)$

A3 Correct image point on $f(l_1)$

Part (b) (ii) **5 marks** **Att 2**

3 (b) (ii)

$s_1 : ax + by + c = 0$ and $s_2 : ax + by + d = 0$ are two parallel lines.

$$f\left(s_1\right): \frac{a}{5}\left(3x' + y'\right) + \frac{b}{5}\left(-2x' + y'\right) + c = 0 \Rightarrow (3a - 2b)x' + (a + b)y' + 5c = 0.$$

$$f\left(s_2\right): \frac{a}{5}\left(3x' + y'\right) + \frac{b}{5}\left(-2x' + y'\right) + d = 0 \Rightarrow (3a - 2b)x' + (a + b)y' + 5d = 0$$

Coefficients of x' and y' match, so these are parallel lines.

OR

Suppose $f(s_1)$ and $f(s_2)$ are not parallel. Then, they have a point in common, say P'.

f is invertible, so let $P = f^{-1}(P')$.

$P' \in f(s_1) \Rightarrow P \in s_1$ and $P' \in f(s_2) \Rightarrow P \in s_2$.

This contradicts $s_1 \parallel s_2$, (unless they are identical, in which case so are their images).

Blunders (-3)

B1 $f(s_1)$ or $f(s_2)$ not in form $px' + qy' + r = 0$ or $y' = mx' + c$

B2 Incorrect matrix

B3 Incorrect matrix multiplication

B4 Fails to finish correctly

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)
A1 One image point correct
A2 Specific case e.g. using $2x–y–1=0$ and $2x–y+k=0$
A3 Effort at image of one line only

Part (b) (iii) **5 marks** **Att 2**

3 (b) (iii)

$$f(l_2): 8x' + y' = k. \quad \therefore A' \text{ is } \left(\frac{k}{8}, 0\right) \text{ and } B' \text{ is } (0, k).$$

$$\text{Area of triangle } A'OB' = \frac{1}{2}\left|\left(\frac{k}{8}\right)(k)\right| = 9.$$

$$\therefore k^2 = 144 \Rightarrow k = \pm 12. \quad \therefore f(l_2): 8x' + y' = \pm 12 \Rightarrow \text{2x - y} \pm \frac{12}{5} = 0$$

Blunders (-3)
B1 One value of k
B2 Error in area formula
B3 Fails to find l_2 from $f(l_2)$

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 A' or B'

Part (b) (iv) **5 marks** **Att 2**

3 (b) (iv)

$$x = \frac{1}{5}(3x' + y') \text{ and } y = \frac{1}{5}(-2x' + y') \text{ and } A'\left(\frac{k}{8}, 0\right), \; B'(0, k).$$

$$\therefore A \text{ is } \left(\frac{3k}{40}, \frac{-2k}{40}\right) \text{ and } B \text{ is } \left(\frac{k}{5}, \frac{k}{5}\right).$$

$$|\angle A'OB'| = 90^\circ.$$

$$\text{Slope } OA = \frac{\frac{-2k}{40}}{\frac{3k}{40}} = -\frac{2}{3} \text{ and slope} OB = \frac{\frac{k}{5}}{\frac{k}{5}} = 1 \Rightarrow OA \text{ is not} \perp \text{to } OB.$$

$$\therefore |\angle AOB| \neq |\angle A'OB'|.$$

Blunders (-3)
B1 A or B incorrect
B2 Error in slope formula
B3 No conclusion or incorrect conclusion

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Effort to find A or B and stops
A2 Effort at finding angle other than required angle
A3 $|\angle A'OB'| = 90^0$

QUESTION 4

Part (a)	**5 marks**	**Att 2**
Part (b)	**30 (10, 10, 10) marks**	**Att (3, 3, 3)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a) **5 marks** **Att 2**

4 (a) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x+\sin x}{3x}\right)$.

Part (a) **5 marks** **Att 2**

4 (a)

$$\lim_{x\to 0}\left(\frac{\sin 2x+\sin x}{3x}\right)=\frac{2}{3}\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)+\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=\frac{2}{3}+\frac{1}{3}=1.$$

OR

$$\lim_{x\to 0}\left(\frac{\sin 2x+\sin x}{3x}\right)=\lim_{x\to 0}\left(\frac{2\sin x\ \cos x+\sin x}{3x}\right)=\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x\ (2\cos\ x\ +1)}{x}\right)=\frac{1}{3}.1.(2+1)=1.$$

OR

$$\lim_{x\to 0}\left(\frac{\sin 2x+\sin x}{3x}\right)=\lim_{x\to 0}\left(\frac{2\sin\frac{3x}{2}\cos\frac{x}{2}}{3x}\right)=\lim_{x\to 0}\left(\frac{\sin\frac{3x}{2}\cos\frac{x}{2}}{\frac{3x}{2}}\right)=1.\cos 0\ =\ 1$$

Blunders (-3)

B1 Error rewriting as sum of two limits

B2 Error in sin2x as a product of two functions

B3 Mishandling $\dfrac{\sin\theta}{\theta}$

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 Correct answer without work

A2 Correct factors

Part (b)	**30 (10, 10, 10) marks**	**Att (3, 3, 3)**

4 (b) Find all the solutions of the equation

$$\sin 2x + \cos x = 0, \text{ where } 0^\circ \le x \le 360^\circ.$$

Transform equation	**10 marks**	**Att 3**
Solve for cos/sin	**10 marks**	**Att 3**
Solutions	**10 marks**	**Att 3**

4 (b)

$$\sin 2x + \cos x = 0$$

$$2\sin x\cos x + \cos x = 0 \;\Rightarrow\; \cos x(2\sin x + 1) = 0.$$

$$\therefore\; \cos x = 0 \text{ or } \sin x = -\tfrac{1}{2}.$$

$$\therefore\; x = 90^\circ, 270^\circ \text{ or } x = 210^\circ, 330^\circ.$$

$$\text{Solution} = \{90^\circ, 210^\circ, 270^\circ, 330^\circ\}.$$

Blunders (-3)

B1 Error in expansion of *sin 2x*
B2 Error in factors
B3 Error in roots
B4 Missing and /or incorrect solutions (to a max of 3)
B5 Solutions outside the range (to a max of 3)

Slips (-1)

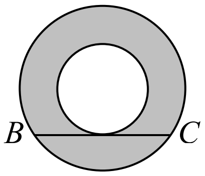
S1 Arithmetic errors

Attempts(3, 3, 3)

A1 *sinx cosx +cosx =0* and stops
A2 One correct angle

Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

4 (c) The diagram shows two concentric circles.
A tangent to the inner circle cuts the outer circle at *B* and *C*, where $|BC| = 2x$.



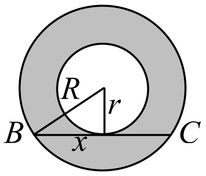
(i) Express the area of the shaded region in terms of *x*.

(ii) In the case where the radius of the outer circle is $2x$, show that the portion of the shaded region that lies below *BC* has area $\left(\frac{2\pi}{3} - \sqrt{3}\right)x^2$.

Part (c) (i) Area in terms of radii 5 marks Att 2

Area in terms of *x* 5 marks Att 2

4 (c) (i)



Let radius of large circle $= R$ and radius of small circle $= r$.

Shaded region $= \pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right)$

But $R^2 = x^2 + r^2 \;\Rightarrow\; R^2 - r^2 = x^2$.

$\therefore$ Shaded region $= \pi x^2$.

Blunders (-3)

B1 Area $= \pi r^2 - \pi R^2$ or $\pi R^2 + \pi r^2$

B2 Incorrect value of *x* for bisected chord

B3 Incorrect use of Pythagoras

Slips (-1)

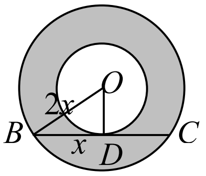
S1 Arithmetic errors

Attempts (2, 2 marks)

A1 Bisector of chord indicated

Part (c) (ii) 5 marks Att 2

4 (c) (ii)



$\sin\angle BOD = \dfrac{x}{2x} = \dfrac{1}{2} \;\Rightarrow\; |\angle BOD| = \dfrac{\pi}{6} \;\Rightarrow\; |\angle BOC| = \dfrac{\pi}{3}$.

$\therefore$ Required area = area of sector *BOC*– area of triangle *BOC*.

$$= \frac{1}{2}r^2\theta - \frac{1}{2}|BC|.|OD|$$

$$= \frac{1}{2}(2x)^2\left(\frac{\pi}{3}\right) - \frac{1}{2}(2x)\left(\sqrt{3}x\right), \quad \left[|OD| = \sqrt{3}x\right]$$

$$= \frac{2x^2\pi}{3} - x^2\sqrt{3} = \left(\frac{2\pi}{3} - \sqrt{3}\right)x^2.$$

Blunders (-3)

B1 $\angle BOC$ incorrect

B2 Incorrect radius substituted into sector formula

B3 Incorrect use of Pythagoras i.e $|OD|$ incorrect

B4 Incorrect conclusion stated or implied

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 Area of sector with some substitution

A2 Required area identified

QUESTION 5

Part (a)	**10 marks**	**Att 3**
Part (b)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**
Part (c)	**25 (10, 10, 5) marks**	**Att (3, 3, 2)**

Part (a) **10 marks** **Att 3**

5 (a) Find the values of x for which $3\tan x = \sqrt{3}$, where $0^\circ \le x \le 360^\circ$.

Part (a) **10 marks** **Att 3**

5 (a)

$$3\tan x = \sqrt{3} \;\Rightarrow\; \tan x = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.$$

$$\therefore\; x = 30^\circ, 210^\circ.$$

Blunders (-3)

B1 Mishandling $\frac{\sqrt{3}}{3}$

B2 Each incorrect angle and/or omitted angle

B3 Each incorrect angle outside the range

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)

A1 One correct angle without work

Part (b) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

5 (b) (i) Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Part (b) (i) Expanding	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

5 (b) (i)

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Blunders (-3)

B1 Error in expanding sin $(A+B)$

B2 Error in expanding $\cos(A+B)$

B3 $\sin A\cos B + \cos A\sin B = \sin(A+B)$ or equivalent not stated

Slips (-1)

S1 Arithmetic error

Attempts (2, 2 marks)

Part (b) (ii) **5 marks** **Att 2**

5 (b) (ii) Show that if $\alpha+\beta=90^\circ$, then $\dfrac{\tan 2\alpha}{\tan 2\beta}=-1$.

Part (b) (ii) **5 marks** **Att 2**

5 (b) (ii)

$$\frac{\tan 2\alpha}{\tan 2\beta}=\frac{\tan 2\alpha}{\tan\left(180^\circ-2\alpha\right)}=\frac{\tan 2\alpha}{-\tan 2\alpha}=-1.$$

Blunders (-3)

B1 Error in $\tan(180^0-2\alpha)$ expansion

B2 Incorrect conclusion

Slips (-1)

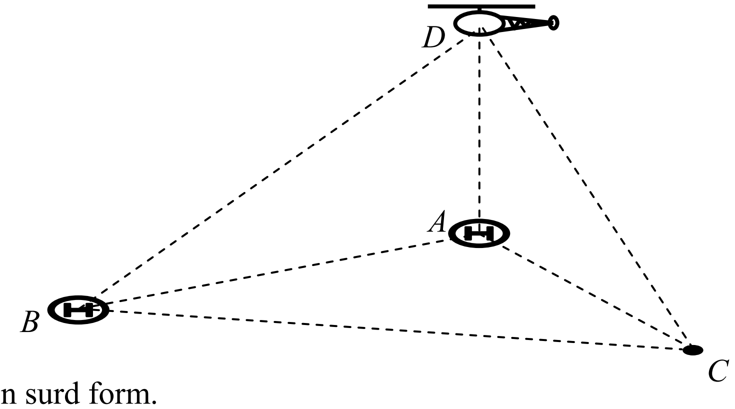
S1 Arithmetic errors

Attempts (2 marks)

A1 $\beta=90^0-\alpha$ or $2\beta=180^0-2\alpha$ and stops

Part (c) **25 (10, 10, 5) marks** **Att (3, 3, 2)**

5 (c) *A* and *B* are two helicopter landing pads on level ground. *C* is another point on the same level ground. $|BC|$ = 800 metres, $|AC|$ = 900 metres, and $|\angle BCA|=60^\circ$. A helicopter is hovering vertically above *A*. A person at C observes the helicopter to have an angle of elevation of 30°.



(i) Find $|AD|$, in surd form.

(ii) Find $|BD|$.

Part (c) (i) **10 marks** **Att 3**

5 (c) (i)

$$\tan 30^\circ = \frac{|AD|}{900} \quad \Rightarrow \quad |AD| = 900\left(\frac{1}{\sqrt{3}}\right) = 300\sqrt{3} \text{ m.}$$

Blunders (-3)

B1 Incorrect use of trigonometric ratio
B2 Answer not in surd form

Slips (-1)

S1 Arithmetic errors
S2 Units omitted or incorrect

Attempts (3 marks)

A1 Identifies relevant right angled triangle

Worthless (0 marks)

W1 Relevant right angled triangle not indicated or implied

Part (c) (ii) $|AB|^2$ **10 marks** **Att 3**

$|BD|$ **5 marks** **Att 2**

5 (c) (ii)

$$|AB|^2 = (800)^2 + (900)^2 - 2(800)(900)\cos 60^\circ$$
$$= 640000 + 810000 - 720000 = 730000$$
$$|BD|^2 = |AB|^2 + |AD|^2 = 730000 + 270000 = 1000000.$$
$$\therefore |\text{BD}| = 1000 \text{ m.}$$

*Accept candidate’s answer from (c)(i)

* If $|AB|^2$ worthless, then attempt at most for remainder of section

Blunders (-3)

B1 Error in cosine formula with substitution
B2 Use of decimals leading to incorrect answer

Slips (-1)

S1 Arithmetic errors
S2 Units omitted or incorrect (apply once only in this section)

Attempts (3, 2 marks)

A1 Cosine formula with some correct substitution

Worthless (0 marks)

W1 Right angle not identified or indicated

QUESTION 6

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

6 **(a)** Two adults and four children stand in a row for a photograph.
How many different arrangements are possible if the four children are between the two adults?

Part (a) **10 marks** **Att 3**

6 (a)

Number of arrangements $= 2! \times 4! = 48$

Blunders (-3)

B1 2!×4!×2!

Attempts (3 Attempts)

A1 4!

A2 2!+4! or 2+4! (with or without further work)

Worthless (0 marks)

W1 6!

Part (b) **20 (10, 10) marks** **Att (3, 3)**

6 (b) **(i)** Solve the difference equation $u_{n+2} - 6u_{n+1} + 8u_n = 0$, where $n \geq 0$, given that $u_0 = 0$ and $u_1 = 4$.

(ii) For what value of n is $u_n = 30(2^n)$?

Part (b) (i) **10 marks** **Att 3**

6 (b) (i)

$$x^2 - 6x + 8 = 0 \;\Rightarrow\; (x-2)(x-4) = 0 \;\Rightarrow\; x = 2 \text{ or } x = 4.$$

$$u_n = l(2)^n + k(4)^n$$

$$u_0 = 0 \;\Rightarrow l + k = 0 \text{ and } u_1 = 4 \Rightarrow 2l + 4k = 4.$$

$$\therefore\ 2l - 4l = 4 \Rightarrow l = -2 \text{ and } k = 2.$$

$$\therefore\ u_n = 2(4)^n - 2(2)^n \;\Rightarrow\; u_n = 2^{2n+1} - 2^{n+1}.$$

Blunders (-3)

B1 Error in setting up quadratic

B2 Error in solving quadratic

B3 Error in general term

B4 Equation in l and k

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)
A1 Substitution into quadratic formula
A2 Equation in l and k

Part (b) (ii) **10 marks** **Att 3**

6 (b) (ii)

$$2^{2n+1} - 2^{n+1} = 30(2^n) \Rightarrow 2^n.2^n.2 - 2^n.2 = 30.2^2 \Rightarrow 2^n.2 - 2 = 30$$

$$\Rightarrow 2^n - 1 = 15 \Rightarrow 2^n = 16 \Rightarrow \therefore n = 4.$$

Blunders (-3)
B1 Error in handling indices

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 $2^{2n+1} = 2^{2n}.2$ or equivalent

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

6 (c) Five cards are drawn together at random from a standard pack of 52 playing cards. Find, in decimal form, correct to two significant figures, the probability that:

(i) all five cards are diamonds

(ii) all five cards are of the same suit

(iii) the five cards are the ace, two, three, four and five of diamonds

(iv) the five cards include the four aces.

Part (c) (i) **5 marks** **Att 2**

6 (c) (i)

$$P(\text{ five diamonds}) = \frac{^{13}C_5}{^{52}C_5} = \frac{1287}{2598960} = 4{\cdot}95\times10^{-4} = 5{\cdot}0\times10^{-4} \text{ or } 0{\cdot}00050$$

Blunders (-3)
B1 Incorrect number of favourable outcomes
B2 Incorrect number of possible outcomes

Slips (-1)
S1 Answer not to two significant figures

Attempts (2 marks)
A1 $\frac{^{13}C_5}{^{52}C_5}$

Part (c) (ii) **5 marks** **Att 2**

6 (c) (ii)

$$P(\text{all same suit}) = P(\text{5 diamonds}) + P(\text{5 hearts}) + P(\text{5 clubs}) + P(\text{5 spades})$$

$$= 4\times\frac{{}^{13}C_5}{{}^{52}C_5} = \frac{5148}{2598960} = 1{\cdot}98\times10^{-3} = 2{\cdot}0\times10^{-3} \text{ or } 0{\cdot}0020$$

Blunders (-3)

B1 Incorrect number of favourable outcomes
B2 Incorrect number of possible outcomes

Slips (-1)

S1 Answer not to two significant figures

Attempts (2 marks)

A1 $4\times\frac{{}^{13}C_5}{{}^{52}C_5}$

Part (c) (iii) **5 marks** **Att 2**

6 (c) (iii)

$$P(\text{ace, 2, 3, 4, 5 of diamonds}) = \frac{{}^{5}C_5}{{}^{52}C_5} = \frac{1}{2598960} = 3{\cdot}84\times10^{-7} = 3{\cdot}8\times10^{-7}$$

or $0{\cdot}00000038$

Blunders (-3)

B1 Incorrect number of favourable outcomes
B2 Incorrect number of possible outcomes

Slips (-1)

S1 Answer not to two significant figures

Attempts (2 marks)

A1 $\frac{{}^{5}C_5}{{}^{52}C_5}$

Part (c) (iv) **5 marks** **Att 2**

6 (c) (iv)

$$P(\text{four aces}) = \frac{{}^{4}C_4\times{}^{48}C_1}{{}^{52}C_5} = \frac{48}{2598960} = 1{\cdot}84\times10^{-5} = 1{\cdot}8\times10^{-5} \text{ or } 0{\cdot}000018$$

Blunders (-3)

B1 Incorrect number of favourable outcomes
B2 Incorrect number of possible outcomes

Slips (-1)

S1 Answer not to two significant figures

Attempts (2 marks)

A1 $\frac{{}^{4}C_4\times{}^{48}C_1}{{}^{52}C_5}$

QUESTION 7

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (10, 10) marks**	**Att (3, 3)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

7 (a) A team of four is selected from a group of seven girls and five boys.

(i) How many different selections are possible?

(ii) How many of these selections include at least one girl?

Part (a) (i) **5 marks** **Att 2**

7 (a) (i)

Number of selections $= {}^{12}C_4 = 495.$

Blunders (-3)

B1 ${}^{7}C_4 + {}^{5}C_4$

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 ${}^{7}C_4$ or ${}^{5}C_4$

Worthless

W1 $\frac{12!}{4!}$

Part (a) (ii) **5 marks** **Att 2**

7 (a) (ii)

Number of selections with no girl $= {}^{5}C_4 = 5.$

Number of selections with at least one girl $= 495 - 5 = 490.$

OR

${}^{7}C_1\,{}^{5}C_3 + {}^{7}C_2\,{}^{5}C_2 + {}^{7}C_3\,{}^{5}C_1 + {}^{7}C_4\,{}^{5}C_0 = 490$

Blunders (-3)

B1 Term omitted

B2 Incomplete answer

Slips (-1)

S1 Arithmetic errors

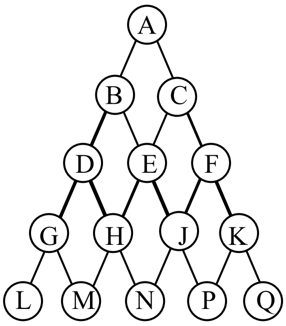
Attempts (2 marks)

A1 ${}^{5}C_4$

A2 ${}^{7}C_1\,{}^{5}C_3$ or equivalent

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

7 (b) A marble falls down from A and must follow one of the path indicated on the diagram. All paths from A to the bottom row are equally likely to be followed.



(i) One of the paths from A to H is A-B-D-H.
List the other two possible paths from A to H.

(ii) Find the probability that the marble passes through H or J.

(iii) Find the probability that the marble lands on N.

(iv) Two marbles fall from A, one after the other, without affecting each other.
Find the probability that they both land at P.

Part (b) (i) **5 marks** **Att 2**

7 (b) (i)
There are two other possible paths: A-B-E-H and A-C-E-H.

Blunders (-3)
B1 One path only

Part (b) (ii) **5 marks** **Att 2**

7 (b) (ii) Paths to J are A-B-E-J, A-C-E-J and A-C-F-J.
$\therefore$ There are 6 paths from A to H or J.
All of the possible paths from A to the GHJK row are:
A-B-D-G, A-B-D-H, A-B-E-H, A-B-E-J, A-C-E-H, A-C-E-J, A-C-F-J, A-C-F-K.
$\therefore$ There are 8 possible paths.
(Or just $2\times2\times2=8$.)
$\therefore$ Probability$=\frac{6}{8}=\frac{3}{4}$.

Blunders (-3)
B1 Number of favourable outcomes incorrect
B2 Number of possible outcomes incorrect

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Favourable and /or all possible outcomes listed correctly

Worthless (0 marks)
W1 Incomplete list of outcomes and stops

Part (b) (iii) **5 marks** **Att 2**

7 (b) (iii)

6 paths to N: ABDHN, ABEHN, ABEJN, ACEHN, ACEJN, ACFJN.

16 possible paths from A to bottom row.

$$\therefore \text{ Probability} = \frac{6}{16} = \frac{3}{8}.$$

Blunders (-3)

B1 Number of favourable outcomes incorrect

B2 Number of possible outcomes incorrect

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 Favourable and /or all possible outcomes listed correctly

Worthless (0 marks)

W1 Incomplete list of outcomes and stops

W2 $\frac{1}{5}$ with or without explanation

Part (b) (iv) **5 marks** **Att 2**

7 (b) (iv)

There are four paths from A to P. $\therefore 4\times4$ outcomes of interest

There are 16 possible paths for each marble. $\therefore 16\times16$ outcomes in total.

$$\therefore \text{ Probability} = \frac{4\times4}{16\times16} = \frac{1}{16}.$$

Blunders (-3)

B1 Number of favourable outcomes incorrect

B2 Number of possible outcomes incorrect

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 Favourable and/or all possible outcomes listed correctly

A2 One marble only

Worthless (0 marks)

W1 $\frac{1}{5}.\frac{1}{5} = \frac{1}{25}$

Part (c) **20 (10, 10) marks** **Att (3, 3)**

7 (c) The real numbers *a*, *b* and *c* have mean μ and standard deviation σ.

(i) Show that the mean of the numbers $\frac{a-\mu}{\sigma}$, $\frac{b-\mu}{\sigma}$ and $\frac{c-\mu}{\sigma}$ is 0.

(ii) Find, with justification, the standard deviation of the numbers

$$\frac{a-\mu}{\sigma}, \frac{b-\mu}{\sigma} \text{ and } \frac{c-\mu}{\sigma}.$$

Part (c) (i) **10 marks** **Att 3**

7 (c) (i)

$$\text{Mean} = \frac{\dfrac{a-\mu+b-\mu+c-\mu}{\sigma}}{3} = \frac{a+b+c-3\mu}{3\sigma} = \frac{3\mu-3\mu}{3\sigma} = 0, \text{ as } \frac{a+b+c}{3} = \mu.$$

Blunders (-3)

B1 $a+b+c \neq 3\mu$ or equivalent

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)

A1 Correct mean of a,b,and c

A2 Expression for mean of $\dfrac{a-\mu}{\sigma}$, $\dfrac{b-\mu}{\sigma}$ and $\dfrac{c-\mu}{\sigma}$

Worthless (0 Marks)

W1 $\dfrac{a-\mu}{\sigma}+\dfrac{b-\mu}{\sigma} + \dfrac{c-\mu}{\sigma}$ and stops

Part (c) (ii) **10 marks** **Att 3**

7 (c) (ii)

The numbers a, b and c have mean μ and standard deviation σ.

$$\therefore\ \sigma = \sqrt{\frac{(a-\mu)^2+(b-\mu)^2+(c-\mu)^2}{3}}.$$

The numbers $\dfrac{a-\mu}{\sigma}$, $\dfrac{b-\mu}{\sigma}$ and $\dfrac{c-\mu}{\sigma}$, with mean = 0, has standard deviation

$$= \sqrt{\frac{\left(\dfrac{(a-\mu)}{\sigma}-0\right)^2+\left(\dfrac{(b-\mu)}{\sigma}-0\right)^2+\left(\dfrac{c-\mu}{\sigma}-0\right)^2}{3}}$$

$$= \frac{1}{\sigma}\sqrt{\frac{(a-\mu)^2+(b-\mu)^2+(c-\mu)^2}{3}} = \frac{1}{\sigma}(\sigma) = 1$$

Blunders (-3)

B1 Error in squaring

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)

A1 Expression for standard deviation correct

QUESTION 8

Part (a)	**15 marks**	**Att 5**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

Part (a) **15 marks** **Att 5**

8 (a) Use integration by parts to find $\int x \sin x \, dx$.

Part (a) **15 marks** **Att 5**

8 (a)

$$\int u dv = uv - \int v du.$$

Let $u = x \Rightarrow du = dx$ and $dv = \int \sin x dx \Rightarrow v = -\cos x$.

$$\therefore \int x \sin x \, dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + \text{constant of integration.}$$

Blunders (-3)

B1 Incorrect differentiation or integration

B2 Incorrect 'parts' formula

Slips (-1)

S1 Arithmetic error

S2 Omits constant of integration

Attempts (5 marks)

A1 One correct assigning in 'parts' formula

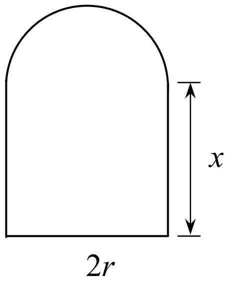
A2 Correct relevant differentiation or integration

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

8 (b) A window is in the shape of a rectangle with a semicircle on top. The radius of the semicircle is r metres and the height of the rectangular part is x metres.
The perimeter of the window is 20 metres.

(i) Use the perimeter to express x in terms of r and π.

(ii) Find, in terms of π, the value of r for which the area of the window is a maximum



Part (b) (i) **5 marks** **Att 2**

8 (b) (i) Perimeter $= 2x + 2r + \pi r = 20 \Rightarrow x = \dfrac{20 - 2r - \pi r}{2}$ metres.

Blunders (-3)
B1 Error in perimeter
B2 Answer not in required form

Slips (-1)
S1 Arithmetic errors
S2 Omits units or incorrect units

Attempts (2 marks)
A1 Expression for perimeter of semicircle
A2 Expression for perimeter of rectangular section of window

Part (b) (ii)Area in terms of *r*	**5 marks**	**Att 2**
Differentiation	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

8 (b) (ii)

Area of window $= A = 2rx + \tfrac{1}{2}\pi r^2.$

$$\therefore\ A = 2r\left(\frac{20-2r-\pi r}{2}\right)+\tfrac{1}{2}\pi r^2 = 20r - 2r^2 - \tfrac{1}{2}\pi r^2.$$

$$\therefore\ \frac{dA}{dr} = 20 - 4r - \pi r.$$

For $\frac{dA}{dr} = 0 \Rightarrow\ 20 - 4r - \pi r = 0$

$\Rightarrow\ r(4+\pi) = 20.$

$$\therefore\ r = \frac{20}{4+\pi}.$$

$\frac{d^2A}{dr^2} = -4-\pi < 0.$ $\therefore$Area of window is a maximum for $r = \frac{20}{4+\pi}$ metres

* If candidate's expression for perimeter in (b)(i) contains square units, then cannot get any further marks in this section

Blunders (-3)
B1 Error in eliminating *x* from expression for area
B2 Error in differentiation
B3 Error in finding *r*

Slips (-1)
S1 Arithmetic errors
S2 Omits units or incorrect units

Attempts (2, 2, 2)
A1 Some correct differentiation
A2 $20 - 4r - \pi r = 0$ and stops

Worthless (0 marks)
W1 Non quadratic expression for area

Part (c) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

8 (c) The Maclaurin series for $\tan^{-1}x$ is $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$

(i) Write down the general term of the series.

(ii) Use the Ratio Test to show that the series converges for $|x|<1$.

(iii) Using the fact that $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$, and taking the first three terms in the Maclaurin series for $\tan^{-1}x$, find an approximation for π.
Give your answer correct to five decimal places.

Part (c) (i) **5 marks** **Att 2**

8 (c) (i)

$$u_n=\frac{x^{2n-1}}{2n-1}(-1)^{n+1}$$

Blunders (-3)

B1 -1 omitted in general term

B2 Incorrect x index in general term

B3 Value of n in denominator does not match index of x in numerator

Slips (-1)

Attempts (2 marks)

A1 One part of general term correct

Part (c) (ii) **5 marks** **Att 2**

8 (c) (ii)

$$\underset{n\to\infty}{\text{Limit}}\left|\frac{u_{n+1}}{u_n}\right|=\underset{n\to\infty}{\text{Limit}}\left|\frac{x^{2n+1}}{2n+1}(-1)^{n+2}\times\frac{2n-1}{x^{2n-1}(-1)^{n+1}}\right|$$

$$=\underset{n\to\infty}{\text{Limit}}\left|\frac{x^2(2n-1)}{2n+1}(-1)\right|=\underset{n\to\infty}{\text{Limit}}\left|\frac{x^2\left(2-\frac{1}{n}\right)}{2+\frac{1}{n}}\right|=x^2.$$

$\therefore$ Convergent for $x^2<1 \quad\Rightarrow\quad$ convergent for $|x|<1$.

*Note: If candidate gets 0 marks in (c)(i) then attempt mark at most in (c)(ii)
If candidate's x index is incorrect in (c)(i), then attempt mark at most in (c)(ii)

Blunders (-3)

B1 Error in u_{n+1}

B2 Error in limits other than slips

B3 $|x^2|$ or $|-x^2|$ mishandled

B4 Incorrect conclusion

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 Ratio test used correctly

Part (c) (iii) **5 marks** **Att2**

8 (c) (iii)

$$\frac{\pi}{4} = 4\left[\frac{1}{5} - \frac{1}{3(5)^3} + \frac{1}{5(5)^5}\right] - \left[\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5}\right]$$

$\therefore\ \pi = 3{\cdot}14162.$

Blunders (-3)

B1 Term omitted in expansion

Slips (-1)

S1 Arithmetic error

Attempts (2 marks)

A1 Correct listing of one series and stops

QUESTION 9

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

Part (a) **10 marks** **Att 3**

9 (a) *Z* is a random variable with standard normal distribution.
Use the tables to find the value of z_1 for which $P(Z \geq z_1) = 0{\cdot}0778$.

Part (a) **10 marks** **Att 3**

9 (a)

$$P(Z \geq z_1) = 0{\cdot}0778 \;\Rightarrow\; 1 - P(Z \leq z_1) = 0{\cdot}0778.$$
$$P(Z \leq z_1) = 0{\cdot}9222 \;\Rightarrow\; z_1 = 1{\cdot}42.$$

Blunders (-3)
B1 Incorrect reading of tables
B2 Incorrect area

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 $P(Z \geq z_1) \;\Rightarrow\; 1 - P(Z \leq z_1)$

Part (b) **20 (10, 10) marks** **Att (3, 3)**

9 (b) A die is biased in such a way that the probability of rolling a six is *p*.
The other five numbers are equally likely. This biased die and a fair die are rolled simultaneously. Show that the probability of rolling a total of 7 is independent of *p*.

Probability of other single outcome	**10 marks**	**Att 3**
Finish	**10 marks**	**Att 3**

9 (b)

Probability of 6 on biased die = *p*
Probability of not 6 on biased die =1-*p*

$\Rightarrow$ probability of any other single outcome (of which there are 5) on die $= \dfrac{1-p}{5}$.

Probability of a total of seven from biased and fair die

$[\text{i.e.} (6,1), (5,2), (4,3), (3,4), (2,5), (1,6)]$

$$= p\left(\frac{1}{6}\right) + \left(\frac{1-p}{5}\right)\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6} + \left(\frac{1-p}{5}\right)\frac{1}{6}$$

$$= \frac{p}{6} + \frac{5}{6}\left(\frac{1-p}{5}\right) = \frac{p+1-p}{6} = \frac{1}{6}.$$

Blunders (-3)
B1 Divisor other than 5
B2 Each term omitted to max of 3
B3 Incorrect or no conclusion written or implied

Slips (-1)
S1 Arithmetic errors

Attempts (3, 3 marks)
A1 Reference to $1-p$
A2 Listing favourable outcomes (must have (6, 1) and at least one other outcome)
A3 One correct term

Part (c) **20 (5, 5, 10) marks** **Att (2, 2, 3)**

9 (c) The mean percentage mark for candidates in the 2010 Leaving Certificate Higher Level Mathematics examination was 67·0%, with a standard deviation of 10·4%. The suggestion that candidates who appealed their results have, on average, similar results to all other candidates, is being investigated. A random sample of candidates who appealed is taken. The mean percentage mark of this sample is 69·3%.

(i) Show that if the sample size was 25, then this result *is not* significant at the 5% level.

(ii) Show that if the sample size was 100, then this result *is* significant at the 5% level.

(iii) What is the smallest sample size for which this result could be regarded as significant at the 5% level?

Part (c) (i) **5 marks** **Att 2**

9 (c) (i)

$$n=25,\quad \mu=67,\quad \sigma=10{\cdot}4,\quad \bar{x}=69{\cdot}3.$$

$$\frac{\bar{x}-\mu}{\sigma_{\bar{x}}}=\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}=\frac{69{\cdot}3-67}{\frac{10{\cdot}4}{\sqrt{25}}}=\frac{2{\cdot}3}{2{\cdot}08}=1{\cdot}105<1{\cdot}96.$$

$\therefore$ Result not significant.

OR

$$\mu-1{\cdot}96\sigma_{\bar{X}}\le\bar{x}\le\mu+1{\cdot}96\sigma_{\bar{X}}$$

$$67-\frac{(1{\cdot}96)(10{\cdot}4)}{\sqrt{25}}\le\bar{x}\le 69{\cdot}3+\frac{(1{\cdot}96)(10{\cdot}4)}{\sqrt{25}}$$

$62{\cdot}9232\le\bar{x}\le 71{\cdot}0768$

Within range $\Rightarrow$ not significant

Blunders (-3)
B1 Error in formula
B2 $\sigma_{\bar{x}}\ne\frac{\sigma}{\sqrt{n}}$
B3 Incorrect or no conclusion implied

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Formula partially substituted

Part (c) (ii) **5 marks** **Att 2**

9 (c) (ii)

$$n = 100,\ \mu = 67,\ \sigma = 10{\cdot}4,\ \bar{x} = 69{\cdot}3.$$

$$\frac{69{\cdot}3 - 67}{\frac{10{\cdot}4}{\sqrt{100}}} = \frac{2{\cdot}3}{1{\cdot}04} = 2{\cdot}211 > 1{\cdot}96.$$

$\therefore$ Result is significant.

Blunders (-3)

B1 Error in formula

B2 $\sigma_{\bar{x}} \neq \frac{\sigma}{\sqrt{n}}$

B3 Incorrect or no conclusion

Slips (-1)

S1 Arithmetic errors

Attempts (2 marks)

A1 Formula partially substituted

Part (c) (iii) **10 marks** **Att 3**

9 (c) (iii)

$$\mu = 67,\ \sigma = 10{\cdot}4\ \ \bar{x} = 69{\cdot}3.$$

$$\frac{69{\cdot}3 - 67}{\frac{10{\cdot}4}{\sqrt{n}}} = \frac{2{\cdot}3\sqrt{n}}{10{\cdot}4} \geq 1{\cdot}96.$$

$$2{\cdot}3\sqrt{n} \geq 1{\cdot}96 \times 10{\cdot}4 \quad \Rightarrow \quad \sqrt{n} \geq 8{\cdot}862.$$

$$\therefore\ \ n > 78{\cdot}55 \quad \Rightarrow \quad n = 79.$$

$\therefore$ Smallest sample size is 79.

Blunders (-3)

B1 Error in formula

B2 $\sigma_{\bar{x}} \neq \frac{\sigma}{\sqrt{n}}$

B3 Incorrect or smallest sample not chosen

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)

A1 Formula partially substituted

QUESTION 10

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**40 (5, 5, 5, 5, 10, 5, 5) marks**	**Att (2, 2, 2, 2, 3, 2, 2)**

Part (a) **10 (5, 5) marks** **Att (2, 2)**

10 **(a)** A Cayley table for the group $(\{a, b, c\},*)$ is shown.

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

(i) Write down the identity element.

(ii) Write down the inverse of each element.

Part (a) (i) **5 marks** **Att 2**

10 (a) (i)

Identity element $= b$.

Attempts (2 marks)

A1 Identity property stated but element not identified

Part (a) (ii) **5 marks** **Att 2**

10 (a) (ii)

$$a^{-1} = c, \quad b^{-1} = b, \quad c^{-1} = a.$$

Blunders (-3)

B1 Inverse of any element omitted

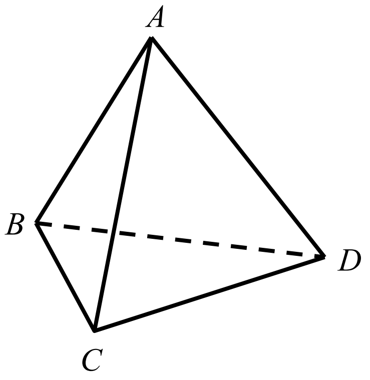
Attempt (2 marks)

A1 $a * a^{-1} = b$

A2 Any correct inverse

Part (b) **40 (5, 5, 5, 5, 10, 5, 5) marks** **Att (2, 2, 2, 2, 3, 2, 2)**

10 (b) A regular tetrahedron has twelve rotational symmetries. These form a group under composition, $\circ$. The symmetries can be represented as permutations of the vertices A, B, C and D.



(i) Write down in permutation form, one element x of order 3, and describe this symmetry geometrically.

(ii) Write down in permutation form, one element y of order 2, and describe this symmetry geometrically

(iii) Show that $x \circ y \neq y \circ x$.

(iv) Let S be the set $\{e,\ x,\ y,\ x \circ y,\ y \circ x,\ x \circ x\}$, where e is the identity transformation. Show that S is not closed under $\circ$.

(v) Let H be a subgroup of G. Let $x \in H$ and $y \in H$. Show that $H = G$.

Part (b) (i) Permutation	**5 marks**	**Att 2**
Description	**5 marks**	**Att 2**

10 (b) (i) Fix one vertex e.g. A

There are eight possible answers, such as:

$$x = \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix}$$

Geometrically, this is a rotation of $\frac{2\pi}{3}$ about the axis AG, where G is the centroid of the triangle BCD.

The other solutions correspond to rotations of $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ about this or similar axes.

Blunders (-3)

B1 Permutation other than order 3

B2 Incomplete geometrical justification

Slips (-1)

S1 Arithmetic errors

Attempts (2, 2 marks)

A1 Incorrect angle of rotation

Part (b) (ii)Permutation	**5 marks**	**Att 2**
Interpretation	**5 marks**	**Att 2**

10 (b) (ii)

There are three possible answers, such as:

$$y = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}.$$

Geometrically, this is a rotation of π about the axis through the mid points of the opposite edges [AD] and [BC].

Blunders (-3)

B1 Incomplete geometrical interpretation

Slips (-1)

S1 Arithmetic errors

Attempt (2, 2 marks)

A1 Reference to π

Part (b) (iii)	**10 marks**	**Att 3**

10 (b) (iii)

$$x \circ y = \begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix}\begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ B & D & C & A \end{pmatrix}.$$

$$y \circ x = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}\begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ D & B & A & C \end{pmatrix}.$$

$\therefore\ x \circ y \neq y \circ x.$

Note: compositions depend on candidate's choice of x and y, but will be unequal in all correct cases.

Blunders (-3)
B1 Error in composition
B2 Incorrect conclusion stated or implied

Slips (-1)
S1 Arithmetic errors

Attempts (3 marks)
A1 $x \circ y$ identified

Part (b) (iv) **5 marks** **Att 2**

10 (b) (iv)

$$(y \circ x)(x \circ y) = \begin{pmatrix} A & B & C & D \\ D & B & A & C \end{pmatrix}\begin{pmatrix} A & B & C & D \\ B & D & C & A \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ B & C & A & D \end{pmatrix} \notin S.$$

$\therefore$ S is not closed.

Note: other correct examples of non-closure exist, and are dependent on candidate's choice of x and y.

Blunders (-3)
B1 Incorrect composition
B2 No conclusion stated or implied

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 At least 2 elements of composition correct

Part (b) (v) **5 marks** **Att 2**

10 (b) (v)

By Lagrange's theorem, any subgroup H of G must be of order 1, 2, 3, 4, 6 or 12.
But H must at least contain the elements $\{e,\ x,\ y,\ x \circ y,\ y \circ x,\ x \circ x\}$.
But by part (iii), this set is not closed. Thus it must contain 12 elements.
Hence $H = G$.

Blunders (-3)
B1 Error in use of Lagrange's Theorem
B2 No reference to issue of non-closure from (iii)

Slips (-1)
S1 Arithmetic errors

Attempts (2 marks)
A1 Definition of a subgroup written or implied.

QUESTION 11

Part (a)	**10 marks**	**Att 3**
Part (b)	**40 (10, 5, 10, 15) marks**	**Att (3, 2, 3, 5)**

Part (a) **10 marks** **Att 3**

11 (a) An ellipse, centre $(0,0)$, has eccentricity $\frac{1}{2}$. One focus is at (2,0). Find the equation of the ellipse.

Part (a) **10 marks** **Att 3**

11 (a)

$$ae = 2 \Rightarrow a(\frac{1}{2}) = 2 \Rightarrow \quad a = 4 \text{ and } b^2 = a^2(1-e^2) \Rightarrow b^2 = 16\left(1-\frac{1}{4}\right) = 12.$$

Ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1.$

Blunders (-3)

B1 Values of a^2 and b^2 found but equation not formed
B2 Error in formula
B3 Mishandling e^2

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)

A1 $a = 4$ and stops

Part (b) **40 (10, 5, 10, 15) marks** **Att (3, 2, 3, 5)**

11 (b)(i) $P(x_1, y_1)$ and $Q\ (x_2, y_2)$ are two distinct points such that $x_1 \le x_2$. If the slope of PQ is $\tan\theta$, and the length of $[PQ]$ is d, express $(x_2 - x_1)$ and $(y_2 - y_1)$ in terms of d and θ.

(ii) Let f be the transformation $(x, y) \to (x', y')$, where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Show that $\dfrac{|f(P)f(Q)|}{|PQ|} = \sqrt{(2\cos\theta + 5\sin\theta)^2 + (3\cos\theta + 4\sin\theta)^2}$.

(iii) Deduce that the ratio of lengths on parallel lines is invariant under f.

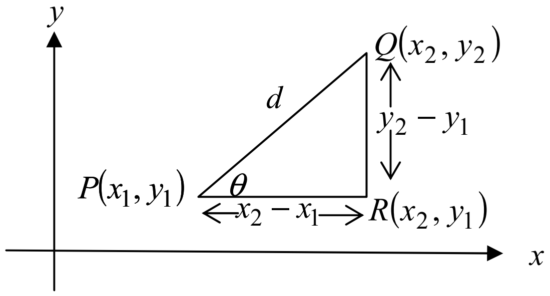
Part (b) (i) **10 marks** **Att 3**

11 (b) (i)

$$|PR| = x_2 - x_1 \quad \text{and} \quad |QR| = y_2 - y_1.$$

$$\cos\theta = \frac{x_2 - x_1}{d} \quad \Rightarrow \quad x_2 - x_1 = d\cos\theta.$$

$$\sin\theta = \frac{y_2 - y_1}{d} \quad \Rightarrow \quad y_2 - y_1 = d\sin\theta.$$



Blunders (-3)

B1 Error in trigonometric formula

B2 $x_2 - x_1 = d\cos\theta$ only

Slips (-1)

S1 Arithmetic errors

Attempts (3 marks)

A1 $\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

Part (b) (ii) $\boldsymbol{\dfrac{|f(P)f(Q)|}{|PQ|}}$ **5 marks** **Att 2**

Finish **10 marks** **Att 3**

11 (b) (ii)

$$f(P) = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x_1 + 5y_1 + 6 \\ 3x_1 + 4y_1 + 1 \end{pmatrix} \quad \text{and} \quad f(Q) = \begin{pmatrix} 2x_2 + 5y_2 + 6 \\ 3x_2 + 4y_2 + 1 \end{pmatrix}.$$

$$\therefore \frac{|f(P)f(Q)|}{|PQ|} = \frac{\sqrt{(2x_2 + 5y_2 + 6 - 2x_1 - 5y_1 - 6)^2 + (3x_2 + 4y_2 + 1 - 3x_1 - 4y_1 - 1)^2}}{d}$$

$$= \frac{\sqrt{[2(x_2 - x_1) + 5(y_2 - y_1)]^2 + [3(x_2 - x_1) + 4(y_2 - y_1)]^2}}{d}$$

$$= \frac{\sqrt{(2d\cos\theta + 5d\sin\theta)^2 + (3d\cos\theta + 4d\sin\theta)^2}}{d}$$

$$= \frac{d\sqrt{(2\cos\theta + 5\sin\theta)^2 + 3(\cos\theta + 4\sin\theta)^2}}{d}$$

$$= \sqrt{(2\cos\theta + 5\sin\theta)^2 + (3\cos\theta + 4\sin\theta)^2}$$

Blunders (-3)

B1 Error in matrix multiplication

B2 Incorrect conclusion

Slips (-1)

S1 Arithmetic errors

Attempts (2, 3 marks)

A1 $f(P)$ or equivalent

A2 Distance formula with some correct substitution for $|f(P)f(Q)|$

Part (b) (iii) **15 marks** **Att 5**

11 (b) (iii)

$[PQ] and [RS]$ are parallel lines

$[PQ]$ and $[RS]$ are mapped to $[f(P)f(Q)]$ and $[f(R)f(S)]$ respectively.

By part **(ii)**, $|f(P)f(Q)| = k|PQ|$, where $k = \sqrt{(2\cos\theta + 5\sin\theta)^2 + (3\cos\theta + 4\sin\theta)^2}$.

Since k depends only on θ, it is the same k for both segments.

$$\therefore \frac{|f(P)f(Q)|}{|f(R)f(S)|} = \frac{k|PQ|}{k|RS|} = \frac{|PQ|}{|RS|}.$$

Blunders (-3)

B1 Fails to justify $|f(R)f(S)| = k|RS|$

B2 No conclusion or incorrect conclusion

Slips (-1)

S1 Arithmetic errors

Attempt (5 marks)

A1 $|f(P)f(Q)| = k|PQ|$

MARCANNA BREISE AS UCHT FREAGAIRT TRÍ GHAEILGE

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronnadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d’iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú **síos**.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéir ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. 198 marc × 5% = 9·9 ⇒ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, ríomhtar an bónas de réir na foirmle [300 – bunmharc] × 15%, agus an marc bónais sin a shlánú **síos**. In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

Bunmharc	Marc Bónais
226	11
227 – 233	10
234 – 240	9
241 – 246	8
247 – 253	7
254 – 260	6
261 – 266	5
267 – 273	4
274 – 280	3
281 – 286	2
287 – 293	1
294 – 300	0